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The Complete Enumeration of Extreme Senary Forms

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THE COMPLETE ENUMERATION OF EXTREME SENARY FORMS

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CONTENTS

Let $f(x_1, \ldots, x_6)$ be a positive definite senary quadratic form of determinant D. Let M be its minimum value for integers $x_1, \ldots, x_6$, not all zero. The form is said to be extreme if, for all infinitesimal variations of the coefficients, M^6/D is maximum. It is proved here for the first time that there are exactly six classes of extreme senary forms, namely, the classes containing the six forms denoted by $\phi_0, \ldots, \phi_4$ and ϕ_6. (Another form ϕ_5 is shown to be only 'perfect', not extreme.) The forms $\phi_0, \phi_1, \phi_2, \phi_4$ are equivalent to A_6, D_6, E_6, E_6^3 in the notation of Coxeter (1951, p. 394); ϕ_3 was discovered simultaneously by M. Kneser and the author (1955); ϕ_6 is new.

Although the analogous forms in fewer variables have been known since 1877, the only previous enumeration of extreme forms in six variables was by Hofreiter (1933), who missed ϕ_3, ϕ_4, ϕ_6, and proposed instead an incorrect form which he called F_4.

1. Introduction

Let $f(\mathbf{x}) = f(x_1, \ldots, x_n) = \Sigma a_{ij} x_i x_j$ be a positive-definite quadratic form of determinant D, and let M be the minimum of $f(\mathbf{x})$ for integral $\mathbf{x} \neq \mathbf{0}$. Suppose f attains its minimum M for the $2s$ sets

$$\mathbf{x} = \pm \mathbf{m}_k = \pm (m_{1k}, \ldots, m_{nk}) \quad (k = 1, \ldots, s);$$

the sets $\mathbf{m}_k$ are then called the *minimal vectors* of f.

f is said to be *perfect* if it is uniquely determined by its minimum and its s minimal vectors, i.e. if the equations

$$\sum_{i,j} a_{ij} m_{ik} m_{jk} = M \quad (k = 1, \ldots, s)$$

have a unique solution for the $N = \frac{1}{2}n(n+1)$ coefficients a_{ij} $(= a_{ji})$; this clearly necessitates $s \geqslant N$.

f is said to be *extreme* if the ratio M^n/D does not increase when the coefficients a_{ij} suffer any sufficiently small variation, i.e. if M^n/D is a local maximum. If M^n/D is an absolute maximum over all positive forms in n variables, f is called *absolutely extreme*, and we set

$$\gamma_n = \max \frac{M}{D^{1/n}}.$$

Clearly an absolutely extreme form is also extreme; Korkine & Zolotareff (1877) showed that also an extreme form is perfect. The properties of being perfect, extreme or absolutely extreme are easily seen to be unaffected by equivalence transformations or by multiplication by a positive constant; it is therefore convenient here to unite in one class all forms equivalent to a (positive) multiple of each other.

The problems of determining all classes of perfect, extreme or absolutely extreme forms in n variables (each of these sets including the succeeding one), or of determining γ_n, have been attacked by many writers. In particular, Korkine & Zolotareff (1877) found all perfect forms for $n \leqslant 5$, and Blichfeldt (1935) evaluated γ_6, γ_7 and γ_8 by a purely arithmetical method. His method, however, is exceedingly complicated and his account very condensed. Mordell (1944) showed that the value of γ_8 could be deduced very simply from that of γ_7.

As regards the case $n = 6$, with which this paper is particularly concerned, the literature appears to contain only one well-established complete result, namely, Blichfeldt's evaluation of γ_6. Korkine & Zolotareff (1877) and Voronoi (1907) did not proceed beyond $n = 5$ in their analysis of perfect forms. Hofreiter (1933) used a geometrical method to find all extreme forms in six variables, but there are some serious errors in his work; one of his forms is not extreme (see Coxeter 1951, p. 394), and, as we shall show, there exist three extreme forms not listed by him. Chaundy (1946) used a method of induction to establish the absolutely extreme forms for all $n \leqslant 10$; although his results are certainly correct for $n \leqslant 6$, it appears that the method cannot be justified and presumably gives a wrong result for $n = 12$ (see Coxeter & Todd 1953). Finally, Coxeter (1951) obtained a large number of classes of extreme forms, which included all known forms for $n \leqslant 8$ and, indeed, a new form for $n = 6$; however, as Coxeter remarks, his method finds extreme forms of particular types only and is not intended to be exhaustive.

The discovery of yet another senary form, made independently by Kneser (1955) and the author (1955), prompted the author to re-examine the whole question of extreme senary forms. The method used is Voronoi's algorithm for perfect forms, discussed in § 2, which suggested itself as being probably the most systematic and least susceptible to error. We establish

Theorem 1. *There are just seven classes of perfect senary forms, represented by the forms (each with minimum* 1)

$$
\begin{aligned}
\phi_0 &= \sum_1^6 x_i^2 + \sum_{i<j} x_i x_j,\\
\phi_1 &= \phi_0 - x_1 x_2,\\
\phi_2 &= \phi_0 - x_1 x_2 - x_1 x_3,\\
\phi_3 &= \phi_0 - \tfrac{1}{2}(x_1 x_2 + x_3 x_4 + x_5 x_6),\\
\phi_4 &= \phi_0 - \tfrac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_3 x_6 + x_4 x_5 + x_4 x_6 + x_5 x_6),\\
\phi_5 &= \phi_0 - \tfrac{1}{2}(x_1 x_2 + x_3 x_4 + x_3 x_5 + x_4 x_5),\\
\phi_6 &= \phi_0 - \tfrac{1}{2}(2x_1 x_2 + x_1 x_3 + x_1 x_6 + x_2 x_5 + x_4 x_6 + 2x_5 x_6).
\end{aligned}
$$

Of these, all except ϕ_5 *are extreme.*

In table 1 are listed the six extreme forms, giving also the symbol used by Coxeter (1951), the number s of minimal vectors and the value of $2^6 D/M^6$ (in decreasing order of this quantity).

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TABLE 1. THE EXTREME SENARY FORMS

form	Coxeter's symbol	s	$(2/M)^6 D$
ϕ_0	A_6	21	7
ϕ_3	—	21	$13.3^3/2^6$
ϕ_6	—	21	$7^3/2^6$
ϕ_1	$B_6 \sim D_6$	30	4
ϕ_4	E_6^3	27	$3^5/2^6$
ϕ_2	E_6	36	3

We see from this that the (essentially unique) absolutely extreme form is ϕ_2; thus, in agreement with Blichfeldt (1935)

$$\gamma_6^6 = \frac{1}{D(\phi_2)} = \frac{64}{3}.$$

Of the above forms not listed in Coxeter (1951), ϕ_3 is discussed in Kneser (1955) and Barnes (1955), while ϕ_5 and ϕ_6 are here given for the first time. It is clear, however, that Voronoi knew the existence* of the perfect non-extreme form ϕ_5, which is the simplest known form of this type.†

2. VORONOI'S ALGORITHM

This section gives an outline of the methods and results of Voronoi (1907), which will be fundamental in all that follows.

In the notation of § 1, we associate with the minimal vectors $\mathbf{m}_k$ of a perfect form $\phi(\mathbf{x})$ the linear forms

$$\lambda_k = \lambda_k(\mathbf{x}) = \sum_{i=1}^{n} m_{ik} x_i \quad (k = 1, \ldots, s), \tag{2·1}$$

and call them the (*associated*) *linear forms of* ϕ. Corresponding to ϕ, we define a region $R = R(\phi)$, in the N-dimensional space of the coefficients a_{ij}, as the set of points satisfying

$$f(\mathbf{x}) = \sum a_{ij} x_i x_j \equiv \sum_{k=1}^{s} \rho_k \lambda_k^2 \quad \text{with} \quad \rho_k \geq 0 \quad (k = 1, \ldots, s). \tag{2·2}$$

In the terminology of Bachmann (1923) (where Voronoi's methods are developed in detail), R is the region with the s edge-forms λ_k^2. R may also be defined as the set of points (a_{ij}) satisfying a certain set of homogeneous linear inequalities

$$\psi_k(a_{ij}) \equiv \sum_{i,j} p_{ij}^{(k)} a_{ij} \geq 0 \quad (k = 1, \ldots, \sigma), \tag{2·3}$$

and so is bounded by σ $(N-1)$-dimensional faces

$$W_k\colon\ \psi_k(a_{ij}) = 0 \quad (k = 1, \ldots, \sigma). \tag{2·4}$$

To each face W_k of R there corresponds a uniquely determined neighbouring region R'_k which has this face in common with R, and which corresponds to a perfect form ϕ'_k distinct‡ from ϕ; we say that ϕ and ϕ'_k are *neighbours* (along the face W_k).

* Voronoi (1907, p. 100), without amplifying the remark in any way, states: 'Ce n'est qu'à partir des formes positives à six variables que j'ai rencontré des formes quadratiques positives qui [sont parfaites] et ne sont pas des formes extrêmes'.

† Some perfect non-extreme forms for all $n \geq 11$, based on the structure of ϕ_3, are given in Barnes (1955).

‡ In the sense that ϕ'_k is not a multiple of ϕ, though it may well be equivalent to a multiple of ϕ. Clearly any multiple of ϕ yields the same region R.

The practical efficiency of Voronoi's method lies largely in the simplicity of the relation between neighbouring forms. Let W be the face $\psi(a_{ij}) = \Sigma p_{ij} a_{ij} = 0$ of R, and define (taking $p_{ij} = p_{ji}$) a quadratic form

$$\psi(\mathbf{x}) = \Sigma p_{ij} x_i x_j. \tag{2·5}$$

Then the neighbour ϕ' of ϕ along the face W is given by

$$\phi'(\mathbf{x}) = \phi(\mathbf{x}) + \rho\psi(\mathbf{x}),$$

where ρ is a uniquely determined positive number. ρ may be found (by a finite process) as the minimum of $(\phi - M)/(-\psi)$ for integral $\mathbf{x}$ with $\psi(\mathbf{x}) < 0$.

The set (R) of regions associated with all perfect forms (ϕ) fills simply the space A of all positive quadratic forms, any two regions either being disjoint or having a common face. The number of classes of perfect forms in n variables is, however, finite, and thus they may be found as follows:

Starting with any perfect form, e.g.

$$\phi_0 = \sum_1^n x_i^2 + \sum_{i<j} x_i x_j,$$

we find all its inequivalent neighbours, discarding any equivalent to ϕ_0; we now find all inequivalent neighbours of these forms, discarding any equivalent to forms already found; and so on. This process terminates when we arrive at a set of inequivalent perfect forms

$$\phi_0, \quad \phi_1, \quad \ldots, \quad \phi_{\tau-1},$$

with the property that any neighbour of any form of the set is equivalent to a form of the set. These τ forms will then be a system of representatives of the different classes of perfect forms in n variables.

The work is much lightened by using the group $\mathfrak{g}$ of automorphs of the perfect form ϕ. If $\mathfrak{T} \in \mathfrak{g}$, then $\mathfrak{T}$ permutes the minimal vectors $\mathbf{m}_k$, and so the contragredient transformation $\mathfrak{T}'^{-1}$ permutes* the linear forms λ_k. Thus the contragredient group $\mathfrak{G}$ leaves R invariant and permutes its faces W_k. If the faces W_k, W_l are equivalent under $\mathfrak{G}$, then the corresponding quadratic forms $\psi_k(\mathbf{x})$, $\psi_l(\mathbf{x})$ are equivalent under $\mathfrak{g}$. Since ϕ is invariant under $\mathfrak{g}$, it follows that the neighbours

$$\phi'_k = \phi + \rho_k \psi_k, \quad \phi'_l = \phi + \rho_l \psi_l$$

are equivalent under $\mathfrak{g}$. *Thus equivalent faces of R yield equivalent neighbours of ϕ.*

In order to enumerate all classes of perfect forms, our basic problem is thus to find the inequivalent faces of R, which is initially defined in terms of its edge-forms λ_k^2. Now a face

$$W\colon\ \psi(f) \equiv \Sigma p_{ij} a_{ij} = 0 \tag{2·6}$$

of R is determined by $N-1$ independent edges; if W contains altogether $t \geqslant N-1$ edges, we shall call W briefly a *t-face of R*. If these t edges are given by, say, $\lambda_1^2, \ldots, \lambda_t^2$, the linear forms $\lambda_1, \ldots, \lambda_t$ will be said to lie on W and $\lambda_{t+1}, \ldots, \lambda_s$ to lie off W. We then have

$$\psi(\lambda_k^2) = 0 \quad (k = 1, \ldots, t), \tag{2·7}$$

$$\psi(\lambda_k^2) > 0 \quad (k = t+1, \ldots, s), \tag{2·8}$$

* Here, and throughout the paper, we waive the distinction between the pair of vectors $\pm\mathbf{m}_k$ and between the associated linear forms $\pm\lambda_k$.

and the complete set of forms of R lying on W is given by

$$f \equiv \sum_{k=1}^{t} \rho_k \lambda_k^2 \quad (\rho_k \geqslant 0).$$

Now (2·6) and (2·7) may be written as

$$\psi(\mathbf{m}_k) = \Sigma p_{ij} m_{ik} m_{jk} = 0 \quad (k = 1, \ldots, t), \tag{2·9}$$

$$\psi(\mathbf{m}_k) = \Sigma p_{ij} m_{ik} m_{jk} > 0 \quad (k = t+1, \ldots, s), \tag{2·10}$$

where $\psi(\mathbf{x})$ is the quadratic form (2·5) corresponding to W. Voronoi proves that, conversely, a set of forms $\lambda_1, \ldots, \lambda_t$ are the forms lying on a face W if and only if (a) the equations (2·9) have rank $N-1$ (so that the ratios of the p_{ij} are uniquely determined); (b) this solution (p_{ij}), with an appropriate sign, satisfies (2·10).

These conditions give a practically efficient method of determining all faces of R, which is used exclusively by Voronoi for forms with $s > N$. When $s = N$ (e.g. for ϕ_0) the problem is trivial; for then (2·2) provides N equations for the ρ_k in terms of the a_{ij}, which have a unique solution (the equations clearly have rank N always, since ϕ is perfect). Since f belongs to R if and only if all $\rho_k \geqslant 0$, this solution provides the N inequalities determining R and hence the N faces of R.

The above simple method when $s = N$ points the way to a more direct determination of the faces, which we now describe:

The condition (2·2) for f to belong to R may be written

$$a_{ij} = \sum_{k=1}^{s} \rho_k m_{ik} m_{jk} \quad (i, j = 1, \ldots, n), \tag{2·11}$$

with $\rho_k \geqslant 0$ $(k = 1, \ldots, s)$. Regarding (2·11) as N equations, of rank N, for $\rho_1, \ldots, \rho_s$, we see that the complete solution will involve $l = s - N$ parameters, say $u_1, \ldots, u_l$, and may be written as

$$\rho_k = L_k(a_{ij}) + M_k(\mathbf{u}), \tag{2·12}$$

where L_k, M_k are linear forms and $\mathbf{u} = (u_1, \ldots, u_l)$. We now have

Lemma 2·1. *The forms $\lambda_{t+1}, \ldots, \lambda_s$ are the forms lying off a face W of R if and only if (a) there exists an essentially unique non-trivial linear relation*

$$\sum_{t+1}^{s} \alpha_k M_k(\mathbf{u}) \equiv 0; \tag{2·13}$$

(b) *the coefficients α_k in this relation satisfy*

$$\alpha_k > 0 \quad (k = t+1, \ldots, s). \tag{2·14}$$

The equation of W is then

$$\psi(a_{ij}) \equiv \sum_{t+1}^{s} \alpha_k L_k(a_{ij}) = 0. \tag{2·15}$$

(i) Suppose first that W is a face of R given by (2·6) and that $\lambda_{t+1}, \ldots, \lambda_s$ are the forms lying off W. Then, by (2·7) and (2·8),

$$\psi(\lambda_k^2) = \Sigma p_{ij} m_{ik} m_{jk} = 0 \quad (k = 1, \ldots, t),$$

$$\alpha_k = \psi(\lambda_k^2) = \Sigma p_{ij} m_{ik} m_{jk} > 0 \quad (k = t+1, \ldots, s).$$

Now for an arbitrary solution ρ_k of (2·11), we have

$$\psi(a_{ij}) \equiv \Sigma p_{ij} a_{ij} = \sum_{k=1}^{s} \rho_k (\Sigma p_{ij} m_{ik} m_{jk}) = \sum_{k=1}^{s} \rho_k \psi(\lambda_k^2),$$

whence

$$\psi(a_{ij}) = \sum_{k=t+1}^{s} \alpha_k \rho_k.$$

(2·12) now gives

$$\psi(a_{ij}) = \sum_{k=t+1}^{s} \alpha_k L_k(a_{ij}) + \sum_{k=t+1}^{s} \alpha_k M_k(\mathbf{u}).$$

The relations (2·13), (2·14) and (2·15) now follow at once.

Thus (2·15) is established, and also the necessity of the conditions (*a*) and (*b*), with the exception of the assertion of uniqueness in (*a*).

(ii) Suppose now that the conditions (*a*) and (*b*) are satisfied, and set

$$\chi(a_{ij}) = \sum_{t+1}^{s} \alpha_k L_k(a_{ij});$$

then, by (2·13) and (2·14),

$$\chi(a_{ij}) = \sum_{t+1}^{s} \alpha_k \rho_k \quad (\alpha_k > 0)$$

for an arbitrary solution $\rho_1, \ldots, \rho_s$ of (2·11). Taking the particular forms $f = \lambda_k^2$ in (2·11) and the obvious solution $\rho_k = 1$, $\rho_l = 0$ $(l \neq k)$, we obtain

$$\chi(\lambda_k^2) = 0 \qquad (k = 1, \ldots, t),$$

$$\chi(\lambda_k^2) = \alpha_k > 0 \quad (k = t+1, \ldots, s).$$

It will now follow, as required, that $\chi(a_{ij}) = 0$ is a face W of R and that $\lambda_{t+1}, \ldots, \lambda_s$ are the forms lying off W, provided that the set $\lambda_1^2, \ldots, \lambda_t^2$ contains $N-1$ independent forms. That this is true is easily seen from the uniqueness of the relation (2·13).

(iii) We can now complete the proof of the necessity of (*a*) and (*b*). Suppose, contrary to assertion, that there exists a further linear relation

$$\sum_{t+1}^{s} \beta_k M_k(\mathbf{u}) \equiv 0,$$

in which the β_k are not proportional to the α_k. Since all $\alpha_k > 0$, $\gamma\alpha_k + \beta_k$ has the sign of γ if $|\gamma|$ is large enough for all $k = t+1, \ldots, s$. Hence there exists a value of γ such that

$$\gamma\alpha_k + \beta_k \geq 0 \quad (k = t+1, \ldots, s),$$

while $\gamma\alpha_k + \beta_k = 0$ for some k; also not all $\gamma\alpha_k + \beta_k$ are zero, since the sets (α_k), (β_k) are not proportional.

We thus have a strictly positive relation between the elements of a proper subset of $M_{t+1}(\mathbf{u}), \ldots, M_s(\mathbf{u})$. Proceeding in this way, we eventually obtain a subset, say $M_{t'+1}(\mathbf{u}), \ldots, M_s(\mathbf{u})$, for which there exists a unique linear relation

$$\sum_{t'+1}^{s} \alpha'_k M_k(\mathbf{u}) \equiv 0, \quad \alpha'_k > 0 \quad (t' > t).$$

But now, by (ii), there is a face W' of R such that the forms lying off W' are $\lambda_{t'+1}, \ldots, \lambda_s$; thus W is a proper subset of W', which is clearly impossible.

Lemma 2·1 provides an efficient method of determining the faces of R, at least for forms for which $s-N$ is small, and we shall use it for all ϕ_i except ϕ_2 and ϕ_4.

Since the total number of faces is often very large (it increases rapidly with $s-N$), we must also have a practical method for recognizing equivalent faces (and choosing a single representative as early as possible). It is clear that if W, W' are faces of R containing respectively forms $\lambda_1, \ldots, \lambda_t$ and $\lambda'_1, \ldots, \lambda'_{t'}$, then W and W' are equivalent if and only if the (unordered) sets $\lambda_1, \ldots, \lambda_t$ and $\lambda'_1, \ldots, \lambda'_{t'}$ are equivalent under $\mathfrak{G}$. Thus, for equivalent faces, we require $t = t'$ and

$$(\lambda_1, \ldots, \lambda_t) \sim (\lambda'_1, \ldots, \lambda'_t), \quad (\lambda_{t+1}, \ldots, \lambda_s) \sim (\lambda'_{t+1}, \ldots, \lambda'_s).$$

Our normal procedure will therefore be as follows:

We build up, step by step, the set $S = (\lambda_{t+1}, \ldots, \lambda_s)$ of forms lying off a face W of R, using the group $\mathfrak{G}$ as far as possible to eliminate equivalent sets at each stage. Thus if, as is usually the case, $\mathfrak{G}$ (regarded as a permutation group on the λ_k) is transitive, then any set S

Table 2. The perfect senary forms and their neighbours

form	inequivalent faces of R	neighbour	form	inequivalent faces of R	neighbour	form	inequivalent faces of R	neighbour
ϕ_0	W_1 (20)	ϕ_1	ϕ_2	W_1 (25)	ϕ_1	ϕ_3	W_1 (20)	ϕ_1
				W_2 (20)	ϕ_1		W_2 (20)	ϕ_2
ϕ_1	W_1 (20)	ϕ_0		W_3 (20)	ϕ_1		W_3 (20)	ϕ_5
	W_2 (20)	ϕ_1		W_4 (28)	ϕ_2			
	W_3 (25)	ϕ_2		W_5 (24)	ϕ_2	ϕ_4	W_1 (20)	ϕ_1
	W_4 (20)	ϕ_2		W_6 (22)	ϕ_2		W_2 (24)	ϕ_2
	W_5 (20)	ϕ_2		W_7 (20)	ϕ_3		W_3 (21)	ϕ_5
	W_6 (20)	ϕ_3		W_8 (24)	ϕ_4			
	W_7 (20)	ϕ_4		W_9 (20)	ϕ_5	ϕ_5	W_1 (20)	ϕ_1
	W_8 (20)	ϕ_5		W_{10} (21)	ϕ_5		W_2 (20)	ϕ_2
				W_{11} (20)	ϕ_6		W_3 (21)	ϕ_2
							W_4 (20)	ϕ_3
							W_5 (21)	ϕ_4
						ϕ_6	W_1 (20)	ϕ_2

is equivalent to one containing any preassigned form, say μ_1. We now consider $\mathfrak{G}(\mu_1)$, the subgroup of $\mathfrak{G}$ which leaves μ_1 invariant. Under $\mathfrak{G}(\mu_1)$, the remaining $s-1$ forms will fall into transitive systems $P_1, \ldots, P_r$, and so any set S is equivalent to one containing one of the pairs

$$(\mu_1, \mu_2^{(1)}), \quad \ldots, \quad (\mu_1, \mu_2^{(r)}),$$

where $\mu_2^{(i)}$ is chosen arbitrarily from P_i. Thus we continue until we reach sets S for which there is a unique strictly positive relation (2·13). The process is considerably simplified by the trivial observation that if S, S' are sets of forms lying off faces W, W' respectively, then S' cannot be equivalent to a proper subset of S.

In order to apply this process we clearly do not need to know the full groups $\mathfrak{G}, \mathfrak{G}(\mu_1), \ldots$, but merely sufficient elements of them to obtain the various transitive systems. The use of too small subgroups would merely result in our finding large numbers of equivalent faces.

In the following seven sections, we shall establish in turn the results exhibited in table 2, the columns of which give respectively: the perfect form ϕ; the equivalent faces W of $R(\phi)$ with the number t of edges of W in brackets; the neighbour* of ϕ along the face W.

* We say, for simplicity, that ϕ' is a neighbour of ϕ if ϕ has a neighbour equivalent to ϕ'.

Finally, we quote Voronoi's fundamental result on extreme forms. A form $f(\mathbf{x}) = \Sigma a_{ij} x_i x_j$ (not necessarily perfect) with associated linear forms $\lambda_k(\mathbf{x})$ ($k = 1, \ldots, s$) is said to be *eutactic* if its adjoint $F(\mathbf{x}) = \Sigma A_{ij} x_i x_j$ is expressible in the form

$$F = \sum_{k=1}^{s} \rho_k \lambda_k^2, \quad \rho_k > 0 \quad (k = 1, \ldots, s). \tag{2·16}$$

Thus a perfect form ϕ is eutactic if its adjoint is an interior point of $R(\phi)$.

Voronoi's theorem may now be expressed as:

$f(\mathbf{x})$ is extreme if and only if it is both perfect and eutactic.

A simple direct proof is given by Kneser (1955); the proof in Voronoi (1907) is also given, in a simpler form, in Bachmann (1923).

3. ϕ_0 AND ITS NEIGHBOURS

Voronoi (1907, pp. 145–149) shows very simply that, for all n, the form

$$\phi_0(\mathbf{x}) = \sum_1^n x_i^2 + \sum_{i<j} x_i x_j, \quad D(\phi_0) = (n+1)/2^n, \quad M(\phi_0) = 1,$$

is perfect and extreme. Its associated linear forms are

$$x_i \quad (i = 1, \ldots, n), \qquad x_i - x_j \quad (1 \leqslant i < j \leqslant n);$$

they number $N = \frac{1}{2}n(n+1)$, and they are all equivalent under $\mathfrak{G}$. Thus $R(\phi_0)$ has N equivalent 20-faces. Taking the representative face

$$W_1(20): \ -a_{12} = 0$$

(which contains all forms except $x_1 - x_2$), we obtain for $n \geqslant 3$ the neighbour

$$\phi_1(\mathbf{x}) = \phi_0(\mathbf{x}) - x_1 x_2,$$

which is not equivalent to ϕ_0 for $n \geqslant 4$.

This verifies table 2 for ϕ_0.

For later use we note here the group $\mathfrak{g}$ of automorphs of ϕ_0. Writing

$$u_0 = -x_1 - \ldots - x_n, \quad u_i = x_i \quad (i = 1, \ldots, n),$$

we have

$$2\phi_0 = \sum_{j=0}^{n} u_j^2,$$

and $\mathfrak{g}$ is the set of $(n+1)!$ permutations of $u_0, u_1, \ldots, u_n$. More precisely, this is the factor group $\mathfrak{g}/(\pm \mathfrak{I})$; here, as elsewhere, we waive the distinction between transformations (minimal vectors, associated linear forms) and their negatives.

4. ϕ_1 AND ITS NEIGHBOURS

Voronoi (1907, pp. 152–160) shows that, for all $n \geqslant 4$, the form

$$\phi_1(\mathbf{x}) = \phi_0(\mathbf{x}) - x_1 x_2, \quad D(\phi_1) = 1/2^{n-2}, \quad M(\phi_1) = 1,$$

is perfect and extreme. Its associated linear forms are

$$x_i \quad (i = 1, \ldots, n),$$
$$x_i - x_j \quad (1 \leqslant i < j \leqslant n, \ (i,j) \neq (1,2)),$$
$$x_1 + x_2 - x_k \quad (k = 3, \ldots, n),$$
$$x_1 + x_2 - x_k - x_l \quad (3 \leqslant k < l \leqslant n),$$

so that $s = n^2 - n$.

All neighbours of ϕ_1 are also shown to be equivalent to one of the following perfect forms (for suitable $\rho > 0$):

$$\phi_1 - \rho x_1 x_3, \tag{4·1}$$

$$\phi_1 + \rho(x_1 x_2 - \delta_{34} x_3 x_4 - \delta_{35} x_3 x_5 - \ldots - \delta_{n-1,n} x_{n-1} x_n), \tag{4·2}$$

where each δ_{ij} $(3 \leqslant i < j \leqslant n)$ is 0 or 1. Also, for $n \leqslant 8$, we have $\rho = 1$ in (4·1), and so the neighbour

$$\phi_2 = \phi_1 - x_1 x_3 = \phi_0 - x_1 x_2 - x_1 x_3 \quad (4 \leqslant n \leqslant 8).$$

For $n = 6$, in particular, the face corresponding to (4·1) is

$$-a_1 a_3 = 0,$$

and there are five forms not lying on this face, namely, $x_1 - x_3$, $x_1 + x_2 - x_3$, $x_1 + x_2 - x_3 - x_l$ $(l = 4, 5, 6)$; this is therefore the face W_2 (25) of the table.

We now examine the forms (4·2) for $n = 6$, so that $N = 21$, $s = 30$. If we set

$$u_1 = x_1 + \ldots + x_6, \quad u_2 = x_1 - x_2, \quad u_i = x_i \quad (i = 3, \ldots, 6), \tag{4·3}$$

then $2\phi_1(\mathbf{x}) = \sum_{i=1}^{6} u_i^2$, and the group $\mathfrak{g}$ of automorphs of $\phi_1(\mathbf{x})$ may be defined as the set of all permutations, with arbitrary changes of sign, of $u_1, \ldots, u_6$; thus $\mathfrak{g}$ is of order $2^6 . 6!$

Using simply the permutations of $x_3, \ldots, x_6$, we see that any form (4·2) is equivalent to one of

$$\phi_1 + \rho x_1 x_2, \tag{4·4}$$

$$\phi_1 + (\rho x_1 x_2 - x_3 x_4), \tag{4·5}$$

$$\phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_3 x_5), \tag{4·6}$$

$$\phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_5 x_6), \tag{4·7}$$

$$\phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6), \tag{4·8}$$

$$\phi_1 + \rho(x_2 x_1 - x_3 x_4 - x_3 x_5 - x_4 x_5), \tag{4·9}$$

$$\phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6), \tag{4·10}$$

$$\phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_5 x_6), \tag{4·11}$$

$$\phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6 - x_5 x_6), \tag{4·12}$$

$$\phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_4 x_6), \tag{4·13}$$

$$\phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_4 x_6 - x_5 x_6). \tag{4·14}$$

Using now the full group $\mathfrak{g}$, we may easily verify that (4·6) is transformed into (4·12) by $u_1 \to u_3$, $u_3 \to u_1$, $u_6 \to -u_6$; (4·5) into (4·8) by $u_1 \to u_5$, $u_3 \to -u_3$, $u_4 \to -u_4$, $u_5 \to u_1$; (4·7) into (4·11) by $u_1 \to u_4$, $u_4 \to u_1$, $u_5 \to -u_5$, $u_6 \to -u_6$; (4·9) into (4·13) by $u_1 \to u_5$, $u_5 \to u_1$, $u_6 \to -u_6$. Hence any form (4·2) is equivalent to one of

$$f_1 = \phi_1 + \rho x_1 x_2,$$

$$f_2 = \phi_1 + \rho(x_1 x_2 - x_3 x_4),$$

$$f_3 = \phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_3 x_5),$$

$$f_4 = \phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_5 x_6),$$

$$f_5 = \phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_5),$$

$$f_6 = \phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_4 x_6),$$

$$f_7 = \phi_1 + \rho(x_1 x_2 - x_3 x_4 - x_3 x_5 - x_3 x_6 - x_4 x_5 - x_4 x_6 - x_5 x_6).$$

(i) Taking $\rho = 1$ in f_1 gives $f_1 = \phi_0$.

(ii) Taking $\rho = 1$ in f_2 gives

$$f_2 = \phi_1 + (x_1x_2 - x_3x_4) = \phi_0 - x_3x_4,$$

which is trivially equivalent to $\phi_1 = \phi_0 - x_1x_2$.

(iii) Taking $\rho = 1$ in f_3 gives

$$f_3 = \phi_1 + x_1x_2 - x_3x_4 - x_3x_5 = \phi_0 - x_3x_4 - x_3x_5,$$

which is trivially equivalent to $\phi_2 = \phi_0 - x_1x_2 - x_1x_3$.

(iv) Taking $\rho = \frac{1}{2}$ in f_4 gives

$$f_4 = \phi_1 + \tfrac{1}{2}(x_1x_2 - x_3x_4 - x_5x_6) = \phi_3.$$

(v) Taking $\rho = \frac{1}{2}$ in f_5 gives similarly $f_5 = \phi_5$.

(vi) Taking $\rho = 1$ in f_6 gives

$$f_6 = \phi_0 - x_3x_4 - x_3x_5 - x_4x_6,$$

which is equivalent to ϕ_2; we have in fact $\phi_2(\mathbf{x}) = f_6(\mathbf{y})$ with

$$x_1 = y_1 + y_2 + y_5 + y_6, \quad x_2 = y_1, \quad x_3 = y_2, \quad x_4 = y_3, \quad x_5 = y_4, \quad x_6 = -y_1 - y_2 - y_3 - y_6.$$

(vii) Taking $\rho = \frac{1}{2}$ in f_7 gives immediately $f_7 = \phi_4$.

Since all the faces corresponding to $f_1, \ldots, f_7$ are easily found to be 20-faces, the table of faces and neighbours for ϕ_1 is established, except for possible equivalence among the eight faces. Since equivalent faces have the same number of edges and have equivalent neighbours, we have only to show that W_4 and W_5 (each with 20 edges and neighbour ϕ_2) are inequivalent. This can be settled directly, but the following method is simpler (and, moreover, does not require any information on the groups $\mathfrak{g}$ and $\mathfrak{G}$):

W_4 has $\psi(\mathbf{x}) = x_1x_2 - x_3x_4 - x_3x_5$ (where in the above $f_3 = \phi_1 + \rho\psi$); the ten linear forms λ_k not lying on W_4 are those for which $\psi(\mathbf{m}_k) > 0$, and are therefore

$$\left.\begin{array}{l} x_3 - x_4, \quad x_3 - x_5, \quad x_1 + x_2 - x_k \quad (k = 3, \ldots, 6), \\ x_1 + x_2 - x_3 - x_6, \quad x_1 + x_2 - x_k - x_l \quad (4 \leqslant k < l \leqslant 6). \end{array}\right\} \tag{4·15}$$

Similarly, W_5 has $\psi(\mathbf{x}) = x_1x_2 - x_3x_4 - x_3x_5 - x_4x_6$ (as in f_6 above), and the λ_k not lying on W_5 are

$$\left.\begin{array}{l} x_3 - x_4, \quad x_3 - x_5, \quad x_4 - x_6, \quad x_1 + x_2 - x_k \quad (k = 3, \ldots, 6), \\ x_1 + x_2 - x_3 - x_6, \quad x_1 + x_2 - x_4 - x_5, \quad x_1 + x_2 - x_5 - x_6. \end{array}\right\} \tag{4·16}$$

Now among the forms (4·15) there are just four linear relations of the type $\lambda_1 + \lambda_2 + \lambda_3 \equiv 0$, namely,

$$(x_3 - x_5) + (x_1 + x_2 - x_3) - (x_1 + x_2 - x_5) = 0,$$
$$(x_3 - x_5) + (x_1 + x_2 - x_3 - x_6) - (x_1 + x_2 - x_5 - x_6) = 0,$$
$$(x_3 - x_4) + (x_1 + x_2 - x_3) - (x_1 + x_2 - x_4) = 0,$$
$$(x_3 - x_4) + (x_1 + x_2 - x_3 - x_6) - (x_1 + x_2 - x_4 - x_6) = 0;$$

while among the forms (4·16) there are five linear relations, namely, the first three of the above and

$$(x_4 - x_6) + (x_1 + x_2 - x_4) - (x_1 + x_2 - x_6) = 0,$$
$$(x_4 - x_6) + (x_1 + x_2 - x_4 - x_5) - (x_1 + x_2 - x_5 - x_6) = 0.$$

Since $\mathfrak{G}$ permutes the λ_i and preserves linear relations among the λ_i, it follows at once that the sets (4·15) and (4·16) are not equivalent under $\mathfrak{G}$, and hence that W_4 and W_5 are not equivalent.

5. ϕ_3 AND ITS NEIGHBOURS

It is shown in Kneser (1955) and Barnes (1955) that

$$\phi_3 = \phi_0 - \tfrac{1}{2}(x_1x_2 + x_3x_4 + x_5x_6), \quad D(\phi_3) = 13.3^3/2^{12},$$

has minimum 1 and is perfect and extreme; and that its twenty-one associated linear forms are

$$x_i \quad (i = 1, \ldots, 6),$$

$$x_i - x_j \quad (1 \leqslant i < j \leqslant 6, \ (i,j) \neq (1,2), (3,4), (5,6)),$$

$$x_1 + x_2 - x_3 - x_4, \quad x_1 + x_2 - x_5 - x_6, \quad x_3 + x_4 - x_5 - x_6,$$

which we denote by λ_i $(i = 1, \ldots, 6)$, μ_j $(j = 1, \ldots, 12)$, ν_k $(k = 1, 2, 3)$ respectively.

Now all permutations of $x_1, \ldots, x_6$ which transform each pair (x_1, x_2), (x_3, x_4), (x_5, x_6) into a pair of this set are clearly elements of $\mathfrak{G}$.* It follows at once that the forms of each set (λ_i), (μ_j), (ν_k) are equivalent under $\mathfrak{G}$.

Since here $s = N = 21$, $R(\phi_3)$ has just twenty-one faces, and at most three inequivalent faces, the forms lying off which may be taken as $\lambda_1 = x_1$, $\mu_1 = x_1 - x_3$, $\nu_1 = x_1 + x_2 - x_3 - x_4$. Solving (2·2) for the ρ_k in terms of the a_{ij}, we obtain the faces

$$\psi_1(a_{ij}) = a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} = 0,$$
$$\psi_2(a_{ij}) = -a_{12} - 2a_{13} - a_{34} + a_{56} = 0,$$
$$\psi_3(a_{ij}) = a_{12} + a_{34} - a_{56} = 0.$$

Thus there are at most three inequivalent neighbours of ϕ_3, given, for suitable $\rho > 0$, by

$$f_1 = \phi_3 + \rho(x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_1x_6),$$
$$f_2 = \phi_3 + \rho(-x_1x_2 - 2x_1x_3 - x_3x_4 + x_5x_6),$$
$$f_3 = \phi_3 + \rho(x_1x_2 + x_3x_4 - x_5x_6).$$

(i) Taking $\rho = \frac{1}{2}$ in f_1, we obtain

$$f_1 = \tfrac{3}{2}x_1^2 + x_2^2 + \ldots + x_6^2 + x_1x_2 + \tfrac{3}{2}x_1x_3 + \tfrac{3}{2}x_1x_4 + \tfrac{3}{2}x_1x_5 + \tfrac{3}{2}x_1x_6$$
$$+ x_2x_3 + x_2x_4 + x_2x_5 + x_2x_6 + \tfrac{1}{2}x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + \tfrac{1}{2}x_5x_6,$$

which is equivalent to ϕ_5; we have, in fact, $\phi_5(\mathbf{x}) = f_1(\mathbf{y})$ with

$$x_1 = y_1 + y_3, \quad x_2 = y_1 + y_4, \quad x_3 = -y_1, \quad x_4 = y_5, \quad x_5 = y_6, \quad x_6 = y_2.$$

(ii) Taking $\rho = \frac{1}{2}$ in f_2 gives

$$f_2 = \phi_0 - x_1x_2 - x_1x_3 - x_3x_4,$$

which is trivially equivalent, under permutation of the variables, to

$$\phi_0 - x_3x_4 - x_3x_5 - x_4x_6;$$

and this last form has been shown in § 4 (vi) to be equivalent to ϕ_2.

(iii) Taking $\rho = \frac{1}{2}$ in f_3 gives

$$f_3 = \phi_0 - x_5x_6 \sim \phi_0 - x_1x_2 = \phi_1.$$

* This subgroup is in fact $\mathfrak{G}/(\pm\mathfrak{I})$, as is shown in Kneser (1955).

The three neighbours of ϕ_3 are thus ϕ_5, ϕ_2 and ϕ_1, and these forms are inequivalent; table 2 for ϕ_3 is therefore established.

6. ϕ_5 AND ITS NEIGHBOURS

The discussion of $$\phi_5(\mathbf{x}) = \phi_0(\mathbf{x}) - \tfrac{1}{2}(x_1x_2 + x_3x_4 + x_3x_5 + x_4x_5) \tag{6·1}$$
is greatly simplified by making the preliminary transformation

$$\mathbf{x} = \mathfrak{T}\mathbf{y} = \tfrac{1}{3}\begin{pmatrix} . & 3 & . & . & . & . \\ . & . & 3 & . & . & . \\ -1 & -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 2 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 3 & . & . & . & . & . \end{pmatrix}\mathbf{y} \tag{6·2}$$

of determinant $\frac{1}{3}$. We obtain

$$2\phi_5(\mathbf{x}) = y_1^2 + y_2^2 - y_2y_3 + y_3^2 + y_4^2 + y_5^2 - y_5y_6 + y_6^2. \tag{6·3}$$

This form has determinant $\frac{9}{16}$, so that $D(\phi_5) = 3^4/2^{10}$.

From (6·2) we see that $\mathbf{x}$ is integral if and only if $\mathbf{y}$ is integral and satisfies

$$\sum_1^6 y_i \equiv 0 \pmod 3. \tag{6·4}$$

On noting that the form $y_3^2 - y_2y_3 + y_3^2$ is itself positive definite and assumes a positive value less than 3 only for $\pm(y_2, y_3) = (1, 0)$, $(0, 1)$ or $(1, 1)$, we can easily show that $M(\phi_5) = 1$ and establish the twenty-two minimal vectors. Using co-ordinates contragredient to those in (6·3), we denote the associated linear forms by

$$\begin{array}{lll}
\lambda_{11} = y_2 - y_5, & \lambda_{12} = y_2 - y_6, & \lambda_{13} = y_2 + y_5 + y_6, \\
\lambda_{21} = y_3 - y_5, & \lambda_{22} = y_3 - y_6, & \lambda_{23} = y_3 + y_5 + y_6, \\
\lambda_{31} = y_2 + y_3 + y_5, & \lambda_{32} = y_2 + y_3 + y_6, & \lambda_{33} = y_2 + y_3 - y_5 - y_6; \\
\mu_{01} = y_1 - y_5, & \mu_{02} = y_1 - y_6, & \mu_{03} = y_1 + y_5 + y_6, \\
\mu_{10} = y_2 - y_4, & \mu_{20} = y_3 - y_4, & \mu_{30} = y_2 + y_3 + y_4; \\
\nu_{01} = y_4 - y_5, & \nu_{02} = y_4 - y_6, & \nu_{03} = y_4 + y_5 + y_6, \\
\nu_{10} = y_1 - y_2, & \nu_{20} = y_1 - y_3, & \nu_{30} = y_1 + y_2 + y_3; \\
\kappa = y_1 - y_4. & &
\end{array}$$

We recover the associated linear forms in x co-ordinates (contragredient to those in (6·1)) by the transformation

$$\mathbf{y} = \mathfrak{T}'\mathbf{x} = \tfrac{1}{3}\begin{pmatrix} . & . & -1 & -1 & -1 & 3 \\ 3 & . & -1 & -1 & -1 & . \\ . & 3 & -1 & -1 & -1 & . \\ . & . & -1 & -1 & -1 & . \\ . & . & 2 & -1 & -1 & . \\ . & . & -1 & 2 & -1 & . \end{pmatrix}\mathbf{x}. \tag{6·5}$$

The adjoint of (6·3) is a multiple of

$$\omega_5(\mathbf{y}) = 3y_1^2+4(y_2^2+y_2y_3+y_3^2)+3y_4^2+4(y_5^2+y_5y_6+y_6^2), \tag{6·6}$$

and the most general expression of $3\omega_5$ in the form

$$\Sigma\rho_{ij}\lambda_{ij}^2+\Sigma\sigma_{0j}\mu_{0j}^2+\Sigma\sigma_{i0}\mu_{i0}^2+\Sigma\tau_{0j}\nu_{0j}^2+\Sigma\tau_{i0}\nu_{i0}^2+\theta\kappa^2 \tag{6·7}$$

(where the suffixes run independently through 1, 2, 3) is given by

$$\rho_{ij}=1, \quad \sigma_{0j}=\sigma_{i0}=\alpha, \quad \tau_{0j}=\tau_{i0}=\beta, \quad \theta=0, \quad \text{with} \quad \alpha+\beta=3. \tag{6·8}$$

The coefficient $\theta = 0$ shows that ϕ_5 *is not eutactic, and hence not extreme.*

We now consider the group $\mathfrak{G}$. We first note that, in (6·5), $\mathbf{x}$ is integral if and only if $3y_1, \ldots, 3y_6$ are integral and congruent modulo 3. Hence a linear transformation of $y_1, \ldots, y_6$ corresponds to an integral unimodular transformation of $x_1, \ldots, x_6$ if and only if it is an integral unimodular transformation of $3y_1, \ldots, 3y_6$ which preserves the relation

$$3y_1 \equiv 3y_2 \equiv \ldots \equiv 3y_6 \pmod 3. \tag{6·9}$$

Now $\mathfrak{G}$, as the contragredient of $\mathfrak{g}$, permutes the associated linear forms. Also, from (6·6) to (6·8), a transformation permuting the linear forms is an automorph of ω_5, and hence belongs to $\mathfrak{G}$, if and only if it (*a*) leaves κ invariant, (*b*) leaves invariant or interchanges the sets

$$(\mu) = (\mu_{01}, \mu_{02}, \mu_{03}, \mu_{10}, \mu_{20}, \mu_{30}), \quad (\nu) = (\nu_{01}, \nu_{02}, \nu_{03}, \nu_{10}, \nu_{20}, \nu_{30}).$$

From this it is easily seen that the following transformations* belong to $\mathfrak{G}$:

$$(y_2, y_3, -y_2-y_3)', \quad (y_5, y_6, -y_5-y_6)', \quad (y_1, y_4)', \quad (y_2, y_5)(y_3, y_6).$$

Thus $\mathfrak{G}$, as a permutation group on the linear forms, has transitive systems

$$(\lambda) = (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{33}), \quad ((\mu), (\nu)), \quad (\kappa).$$

To determine the faces of $R(\phi_5)$ we use the method of lemma 2·1, and begin by solving the twenty-one equations obtained by equating $\Sigma a_{ij}x_ix_j$ to the form (6·7) (where the linear forms are written in x co-ordinates to avoid later complications). It suffices to give the following portion of the solution, which involves a single parameter u:

$$\begin{aligned}
\theta &= \sum_1^6 a_{i6},\\
2\rho_{31} &= a_{12}-a_{34}-a_{35}+a_{45},\\
2\sigma_{10} &= 2\sum_1^6 a_{1i}-a_{12}+a_{34}+a_{35}+a_{45}-2u,\\
2\sigma_{30} &= -a_{12}+a_{34}+a_{35}+a_{45}-2u,\\
\sigma_{03} &= -a_{56}-u,\\
2\tau_{01} &= a_{12}+2a_{13}+2a_{23}+2a_{33}+a_{34}-a_{35}-a_{45}+2a_{36}+2u,\\
2\tau_{20} &= -a_{26}+u,\\
\tau_{30} &= u.
\end{aligned}$$

* For convenience we use here, and also in §§8 and 9, the notation $(a, b, \ldots)'$ for the full permutation group with carrier $a, b, \ldots$, and the usual cycle notation for its elements.

We obtain at once the faces $\theta = 0$, $\rho_{31} = 0$, i.e.

$$W_5\,(21):\ \psi_5(a_{ij}) = \sum_{i=1}^{6} a_{i6} = 0,$$

$$W_3\,(21):\ \psi_3(a_{ij}) = a_{12} - a_{34} - a_{35} + a_{45} = 0,$$

each with twenty-one edges; the forms lying off them being respectively κ and λ_{31}.

Since all λ_{ij} are equivalent under $\mathfrak{G}$, R has nine faces all equivalent to W_3. Let now S be a set of forms determining (lying off) a face W not equivalent to W_5, W_3. Then S can contain only the forms (μ), (ν). Since these are all equivalent under $\mathfrak{G}$, we need consider only sets containing ν_{30}. From the above solution, giving $\tau_{30} = u$, ν_{30} does not determine a face, and S contains just one more form.

Now $\mathfrak{G}(\nu_{30})$, the subgroup of $\mathfrak{G}$ leaving ν_{30} invariant, certainly contains $(y_2, y_3)'$ and $(y_5, y_6, -y_5-y_6)'$, so that, under $\mathfrak{G}(\nu_{30})$,

$$\mu_{01} \sim \mu_{02} \sim \mu_{03}, \quad \mu_{10} \sim \mu_{20}, \quad \nu_{01} \sim \nu_{02} \sim \nu_{03}, \quad \nu_{10} \sim \nu_{20}.$$

Hence S is equivalent to one of the sets

$$(\nu_{30}, \mu_{03}), \quad (\nu_{30}, \mu_{10}), \quad (\nu_{30}, \mu_{30}), \quad (\nu_{30}, \nu_{01}), \quad (\nu_{30}, \nu_{20}).$$

The above solution shows that the first three of these yield the faces:

$$W_4\,(20):\ \psi_4(a_{ij}) = \tau_{30} + \sigma_{03} = -a_{56} = 0,$$

$$W_2\,(20):\ \psi_2(a_{ij}) = 2\tau_{30} + 2\sigma_{10} = 2\sum_{1}^{6} a_{1i} - a_{12} + a_{34} + a_{35} + a_{45} = 0,$$

$$W_1\,(20):\ \psi_1(a_{ij}) = 2\tau_{30} + 2\sigma_{30} = -a_{12} + a_{34} + a_{35} + a_{45} = 0.$$

The remaining two sets do not determine faces, since the corresponding linear dependence relations clearly cannot have positive coefficients.

We have thus shown that all neighbours of ϕ_5 are equivalent (with appropriate $\rho > 0$) to one of

$$f_1 = \phi_5 + \rho(-x_1x_2 + x_3x_4 + x_3x_5 + x_4x_5),$$
$$f_2 = \phi_5 + \rho(2x_1^2 + x_1x_2 + 2x_1x_3 + 2x_1x_4 + 2x_1x_5 + 2x_1x_6 + x_3x_4 + x_3x_5 + x_4x_5),$$
$$f_3 = \phi_5 + \rho(x_1x_2 - x_3x_4 - x_3x_5 + x_4x_5),$$
$$f_4 = \phi_5 + \rho(-x_5x_6),$$
$$f_5 = \phi_5 + \rho(x_1x_6 + x_2x_6 + x_3x_6 + x_4x_6 + x_5x_6 + x_6^2).$$

(i) Taking $\rho = \frac{1}{2}$ in f_1 gives $\quad f_1 = \phi_0 - x_1x_2 = \phi_1$.

(ii) Taking $\rho = \frac{1}{2}$ in f_2 gives

$$f_2 = \phi_0 + x_1(x_1 + x_3 + x_4 + x_5 + x_6) \sim \phi_0 - x_1x_2 - x_1x_3 = \phi_2$$

under the transformation $x_3 \to -x_1 - x_2 - x_3 - x_4 - x_5 - x_6$ (which is an automorph of ϕ_0).

(iii) Taking $\rho = \frac{1}{2}$ in f_3 gives

$$f_3 = \phi_0 - x_3x_4 - x_3x_5 \sim \phi_0 - x_1x_2 - x_1x_3 = \phi_2.$$

(iv) Taking $\rho = \frac{1}{2}$ in f_4 gives

$$f_4 = \phi_0 - \tfrac{1}{2}(x_1x_2 + x_3x_4 + x_3x_5 + x_4x_5 + x_5x_6) \sim \phi_0 - \tfrac{1}{2}(x_1x_2 + x_3x_4 + x_3x_5 + x_3x_6 + x_5x_6)$$

by a permutation of variables. This last form was shown in § 4 (where it is the form (4·11) with $\rho = \frac{1}{2}$) to be equivalent to ϕ_3.

(v) Taking $\rho = \frac{1}{2}$ in f_5 gives

$$f_5 = \phi_0 - \tfrac{1}{2}(x_1x_2+x_3x_4+x_3x_5+x_4x_5) + \tfrac{1}{2}x_6(x_1+x_2+x_3+x_4+x_5+x_6) \sim \phi_4.$$

under the transformation $x_1 \to x_1+x_6$, $x_2 \to x_2+x_6$, $x_3 \to x_3$, $x_4 \to x_4$, $x_5 \to x_5$, $x_6 \to -x_6$ (which is just $u_6 \to -u_6$ in the notation of § 4 (4·3)).

Table 2 for ϕ_5 is now completely established, the inequivalence of the five given faces being trivial.

7. ϕ_6 AND ITS NEIGHBOURS

We have

$$\phi_6(\mathbf{x}) = x_1^2+\ldots+x_6^2+\tfrac{1}{2}x_1x_3+x_1x_4+x_1x_5+\tfrac{1}{2}x_1x_6+x_2x_3+x_2x_4$$
$$+\tfrac{1}{2}x_2x_5+x_2x_6+x_3x_4+x_3x_5+x_3x_6+x_4x_5+\tfrac{1}{2}x_4x_6. \quad (7·1)$$

It is not difficult to show, by a direct argument, that we have here $M = 1$, $D = 7^3/2^{12}$, $s = N = 21$. We can, however, simplify the analysis and see the structure of this new form more clearly by making the preliminary transformation

$$\mathbf{x} = \mathfrak{T}\mathbf{y} = \tfrac{1}{7}\begin{pmatrix} 3 & 6 & 2 & 5 & 1 & 4 \\ 5 & 3 & 1 & 6 & 4 & 2 \\ 4 & 1 & 5 & 2 & -1 & 3 \\ -4 & -1 & 2 & -2 & 1 & -3 \\ -1 & -2 & -3 & -4 & 2 & 1 \\ -6 & -5 & -4 & -3 & -2 & -1 \end{pmatrix}\mathbf{y}, \quad (7·2)$$

of determinant $\frac{1}{7}$. We then obtain

$$4\phi_6(\mathbf{x}) = 2\sum_1^6 y_i^2 + 2\sum_{i<j} y_iy_j = 2\phi_0(\mathbf{y}).$$

Following Voronoi's analysis of ϕ_0, we define y_0 by

$$\sum_{i=0}^{6} y_i = 0, \quad (7·3)$$

whence
$$4\phi_6(\mathbf{x}) = 2\phi_0(\mathbf{y}) = \sum_{i=0}^{6} y_i^2. \quad (7·4)$$

Since $2\phi_0$ has determinant 7, we have at once $D(\phi_6) = 7^3/2^{12}$.

From (7·2) and the inverse relation

$$\mathbf{y} = \mathfrak{T}^{-1}\mathbf{x} = \begin{pmatrix} -1 & . & . & -1 & . & -1 \\ 1 & -1 & -1 & . & . & -1 \\ . & . & 1 & 1 & . & . \\ . & 1 & . & . & -1 & 1 \\ . & 1 & . & 1 & . & . \\ 1 & . & 1 & . & . & 1 \end{pmatrix}\mathbf{x},$$

we see that $\mathbf{x}$ is integral if and only if $\mathbf{y}$ is integral and satisfies

$$y_1+2y_2+3y_3+4y_4+5y_5+6y_6 \equiv 0 \pmod 7,$$

i.e.
$$\sum_{i=0}^{6} iy_i \equiv 0 \pmod 7. \quad (7·5)$$

To determine the minimum and minimal vectors of ϕ_6, we have now to consider

$$2\phi_0(\mathbf{y}) = \sum_{i=0}^{6} y_i^2$$

for integral y_i subject to (7·3) and (7·5). Now this form takes only even values. It takes the value 2 only when some two y_i are 1, -1 and the rest are zero; and clearly none of these sets satisfies (7·5). It takes the value 4 only when some four y_i are 1, 1, -1, -1 and the remaining three are zero; since some of these sets satisfy (7·5), we have $M(\phi_6) = 1$. Now a set $y_a = y_b = 1$, $y_c = y_d = -1$ with $2\phi_0(\mathbf{y}) = 4$ satisfies (7·5) when

$$a+b \equiv c+d \pmod 7.$$

Denoting such a set, for convenience, by *abcd*, we find the twenty-one distinct minimal vectors:

$$\left.\begin{array}{l} 0126,\ 0135,\ 0236,\ 0245,\ 0312,\ 0346,\ 0413,\ 0456,\ 0514,\ 0523,\ 0615, \\ 0624,\ 1246,\ 1356,\ 1423,\ 1524,\ 1625,\ 1634,\ 2534,\ 2635,\ 3645. \end{array}\right\} \quad (7\cdot6)$$

Using (7·2), we find the minimal vectors in $\mathbf{x}$ co-ordinates, and the associated linear forms (in co-ordinates contragredient to those in (8·1)) are

$$\left.\begin{array}{l} x_1,\quad x_2,\quad x_3,\quad x_4,\quad x_5,\quad x_6,\quad x_1-x_4,\quad x_1-x_5,\quad x_2-x_3, \\ x_2-x_4,\quad x_2-x_6,\quad x_3-x_4,\quad x_3-x_5,\quad x_3-x_6,\quad x_4-x_5, \\ x_1+x_2-x_4,\quad x_3-x_5-x_6,\quad x_1+x_2-x_4-x_6, \\ x_1+x_2-x_5-x_6,\quad x_1+x_3-x_5-x_6,\quad x_1+x_2+x_3-x_4-x_5-x_6. \end{array}\right\} \quad (7\cdot7)$$

From this it is easily verified that ϕ_6 *is perfect.*

To establish that ϕ_6 is eutactic, and hence extreme, it is more convenient to work with $\mathbf{y}$ co-ordinates contragredient to those in (7·4). The associated linear forms are then

$$\lambda_i(\mathbf{y}) = y_a + y_b - y_c - y_d,$$

where a, b, c, d take the twenty-one values given in (7·6) and we define $y_0 = 0$.

Now $$\sum_{i=1}^{21} \lambda_i^2(\mathbf{y}) = 12 \sum_0^6 y_i^2 - 4 \sum_{i<j} y_i y_j,$$

since, in (7·6), each suffix occurs twelve times, and each pair of suffixes occurs twice as (a, b) and four times as (a, c) (or (a, d), etc.). Since $y_0 = 0$ and the adjoint of $\phi_0(\mathbf{y})$ is a multiple of $6 \sum_1^6 y_i^2 - 2 \sum_{i<j} y_i y_j$, it follows that the adjoint of $\phi_6(\mathbf{x})$ is a multiple of

$$\omega_6 = \sum_{i=1}^{21} \lambda_i^2.$$

Thus ϕ_6 *is eutactic, and hence extreme.*

We now consider the group $\mathfrak{g}$ of automorphs of ϕ_6 (which is more simply described than the contragredient group $\mathfrak{G}$). As we noted in § 3, the group $\mathfrak{g}(\phi_0)$ of automorphs of $\phi_0(\mathbf{y})$ has order 2·7! and is generated by $\pm\mathfrak{J}$ and all permutations of $y_0, y_1, \ldots, y_6$. We may thus describe a subgroup $\mathfrak{g}_6$ of $\mathfrak{g}$ as the set of permutations of $y_0, \ldots, y_6$ which leave the condition (7·5) invariant.

Now (7·5) may be presented in just forty-two different forms, obtained by one or more of the operations: (*a*) multiplying by any number prime to 7; (*b*) substituting for any y_i in terms of the remaining six from (7·3). A permutation of $y_0, \ldots, y_6$ therefore belongs to $\mathfrak{g}$ if it transforms (7·5) into one of the forty-two equivalent forms. From the forms

$$0y_0+5y_1+3y_2+y_3+6y_4+4y_5+2y_6 \equiv 0 \pmod 7,$$
$$6y_0+0y_1+y_2+2y_3+3y_4+4y_5+5y_6 \equiv 0 \pmod 7,$$

obtained from (7·5) by multiplying by 5 and by substituting $y_1 = -y_0-y_2-\ldots-y_6$, respectively, we obtain the elements

$$\mathfrak{U} = (y_1y_3y_2y_6y_4y_5), \quad \mathfrak{V} = (y_0y_1y_2y_3y_4y_5y_6)$$

of $\mathfrak{g}_0$ (written in cycle notation). Since $\mathfrak{U}$ has order 6 and $\mathfrak{V}$ order 7, g_0 is thus the metacyclic group of order 42 generated by $\mathfrak{U}$ and $\mathfrak{V}$.

This subgroup $\mathfrak{g}_0$ of $\mathfrak{g}$ suffices to show that $\mathfrak{g}$ *is transitive on the minimal vectors*. For the vectors in each of the following sets are clearly equivalent under $\{\mathfrak{V}\}$:

(0126, 1230, 2341, 3452, 4563, 5604, 6015),
(0135, 1246, 2350, 3461, 4502, 5613, 6024),
(0236, 1340, 2451, 3562, 4603, 5014, 6125)

(these being essentially identical with (8·6)); while under $\{\mathfrak{U}\}$ we have

$$0126 \sim 0364 (=4603) \sim 0245.$$

Hence $\mathfrak{G}$ is transitive on the associated linear forms. Since $s = N = 21$, $R(\phi_6)$ has twenty-one equivalent faces, which are most simply found by solving for ρ_k from the relation

$$\Sigma a_{ij}x_ix_j = \Sigma \rho_k \lambda_k^2(\mathbf{x}).$$

The face of R not containing $x_3-x_5-x_6$ is thus found to be

$$W_1\,(20):\ \psi_1(a_{ij}) = -a_{13}+a_{16}+a_{25}+a_{46}+2a_{56} = 0,$$

and the corresponding neighbour is

$$\phi_6(\mathbf{x})+\tfrac{1}{2}(-x_1x_3+x_1x_6+x_2x_5+x_4x_6+2x_5x_6) = \phi_2(\mathbf{x}).$$

This establishes table 2 for ϕ_6.

8. ϕ_4 AND ITS NEIGHBOURS

We have

$$\phi_4(\mathbf{x}) = \phi_0(\mathbf{x})-\tfrac{1}{2}(x_1x_2+x_3x_4+x_3x_5+x_3x_6+x_4x_5+x_4x_6+x_5x_6). \tag{8·1}$$

Applying the transformation

$$\mathbf{x} = \mathfrak{T}\mathbf{y} = \tfrac{1}{3}\begin{pmatrix} 1 & . & -1 & -1 & -1 & -1 \\ . & 1 & -1 & -1 & -1 & -1 \\ . & . & . & 1 & 1 & 1 \\ . & . & 1 & . & 1 & 1 \\ . & . & 1 & 1 & . & 1 \\ . & . & 1 & 1 & 1 & . \end{pmatrix}\mathbf{y} \tag{8·2}$$

of determinant $-1/3^5$, we obtain

$$36\phi_4 = 4\sum_1^6 y_i^2+2\sum_{i<j} y_iy_j = (\Sigma y_i)^2+3\Sigma y_i^2. \tag{8·3}$$

From (8·2), $\mathbf{x}$ is integral when and only when $\mathbf{y}$ is integral and

$$y_1 \equiv y_2 \equiv \ldots \equiv y_6 \pmod 3. \tag{8·4}$$

From (8·3) and (8·4) we find easily that $M(\phi_4) = 1$, $s = 27$, and that the minimal vectors, in $\mathbf{y}$ co-ordinates, are

$$(3, 0, 0, \ldots, 0)', \quad (-2, 1, 1, \ldots, 1)', \quad (-2, -2, 1, 1, 1, 1)',$$

where the prime denotes all permutations of the co-ordinates. (ϕ_4 is equivalent to $\frac{3}{2}E_6^3$ of Coxeter (1951, p. 439), as may be verified directly from these results.) In co-ordinates contragredient to those in (8·3), we denote the twenty-seven associated linear forms by

$$\left.\begin{aligned} \lambda_i &= 3y_i & &(i = 1, \ldots, 6), \\ \mu_i &= -\sum_1^6 y_k + 3y_i & &(i = 1, \ldots, 6), \\ \nu_{ij} &= \sum_1^6 y_k - 3y_i - 3y_j & &(1 \leqslant i < j \leqslant 6). \end{aligned}\right\} \tag{8·5}$$

It is now easily verified that ϕ_4 is perfect. That it is eutactic, and hence extreme, follows from the fact that the adjoint of (8·3) is a multiple of

$$12(4\Sigma y_i^2 - \sum_{i<j} y_i y_j) = \Sigma\lambda_i^2 + \Sigma\mu_i^2 + \Sigma\nu_{ij}^2.$$

By a now familiar argument, $\mathfrak{G}$ is the group of linear transformations of $x_1, \ldots, x_6$ which permute the linear forms (8·5); here $\mathbf{x}$ and $\mathbf{y}$ are connected by the relation

$$\mathbf{y} = \mathfrak{T}'\mathbf{x}$$

contragredient to (8·2). We shall need the following elements of $\mathfrak{G}$:

$\mathfrak{P}$: all permutations of $y_1, \ldots, y_6$;

$\mathfrak{U}$: $3y_i \to \sum_1^6 y_k - 3y_i \quad (i = 1, \ldots, 6)$;

$\mathfrak{V}_{123}$: $y_i \to y_i \quad (i = 1, 2, 3), \quad 3y_4 \to \sum_1^6 y_k - 3y_5 - 3y_6,$

$3y_5 \to \sum_1^6 y_k - 3y_4 - 3y_6, \quad 3y_6 \to \sum_1^6 y_k - 3y_4 - 3y_5.$

Elements of $\mathfrak{P}$ effect the same permutation on the suffixes of the forms (8·5). $\mathfrak{U}$ leaves all ν_{ij} invariant and interchanges λ_i and μ_i $(i = 1, \ldots, 6)$. $\mathfrak{V}_{123}$ interchanges the elements of each pair λ_4, ν_{56}; λ_5, ν_{46}; λ_6, ν_{45}; μ_1, ν_{23}; μ_2, ν_{13}; μ_3, ν_{12}; and leaves the remaining fifteen forms invariant. Multiplying $\mathfrak{V}_{123}$ by suitable elements of $\mathfrak{P}$, we obtain a corresponding transformation $\mathfrak{V}_{ijk}$ for any indices i, j, k.

Before establishing some necessary properties of $\mathfrak{G}$, we make the following definitions (which will be used also in § 9):

Two linear forms κ_1, κ_2 of the set (8·5) are said to *combine* if there exists another form κ_3 which is a linear combination of them; we then say also that κ_2 *combines with* κ_1.

All pairs of combining forms are easily found to be

$$\lambda_i, \ \mu_j; \quad \lambda_i, \ \nu_{ij}; \quad \mu_i, \ \nu_{ij}; \quad \nu_{ij}, \ \nu_{kl};$$

the corresponding linear relations are

$$\lambda_i + \mu_j + \nu_{ij} = 0, \quad \nu_{ij} + \nu_{kl} + \nu_{pq} = 0. \tag{8·6}$$

(Here all suffixes are supposed distinct.)

LEMMA 8·1. (i) *$\mathfrak{G}$ is transitive on the associated linear forms.*

(ii) *Under $\mathfrak{G}(\lambda_1)$ (the subgroup of $\mathfrak{G}$ leaving λ_1 invariant), all ten forms combining with λ_1 are equivalent, and all sixteen forms not combining with λ_1 are equivalent.*

Proof. (i) Transitivity is easily established from the subgroup of $\mathfrak{G}$ generated by $\mathfrak{P}$, $\mathfrak{U}$, $\mathfrak{V}_{123}$.

(ii) $\mathfrak{G}(\lambda_1)$ contains all permutations of $y_2, \ldots, y_6$ and $\mathfrak{V}_{123}$, from which follows the equivalence of the forms in each of the sets

$$(\mu_2, \ldots, \mu_6, \nu_{12}, \ldots, \nu_{16}), \quad (\lambda_2, \ldots, \lambda_6, \mu_1, \nu_{23}, \nu_{24}, \ldots, \nu_{56}).$$

The forms of the first set combine with λ_1, those of the second set do not.

We now turn to the faces W of $R(\phi_4)$; we will find it most convenient to combine the two methods outlined in § 2. We denote generally by $S = (\kappa_1, \ldots, \kappa_r)$ the set of forms lying off a face W, so that S determines W.

Following the method of lemma 2·1, we have to obtain the linear forms, denoted by $M_k(\mathbf{u})$ in (2·12), which give the general solution of

$$\sum_1^6 \rho_i \lambda_i^2 + \sum_1^6 \sigma_i \mu_i^2 + \sum_{i<j} \tau_{ij} \nu_{ij}^2 \equiv 0.$$

Such a solution is given by

$$\left.\begin{aligned} &\left.\begin{aligned} \rho_i &= 3u_i \\ \sigma_i &= -\sum_1^6 u_k + 3u_i \end{aligned}\right\} \quad (i = 1, \ldots, 6), \\ &\tau_{ij} = \sum_1^6 u_k - 3u_i - 3u_j \quad (1 \leqslant i < i \leqslant 6); \end{aligned}\right\} \tag{8·7}$$

for these forms have rank $6 = s - N$, and are easily verified to provide a solution. Comparing (8·5) and (8·7), we obtain from lemma 2·1:

LEMMA 8·2. *A set S of forms* (8·5) *determines a face W of R if and only if the forms* (*with signs as in* (8·5)) *possess a unique linear relation which has positive coefficients.*

We obtain by inspection the following three faces:

W_1 (20): $\quad S = (\mu_1, \mu_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \nu_{12}),$

with relation $\quad 2\mu_1 + 2\mu_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \nu_{12} = 0;$

W_2 (24): $\quad S = (\lambda_1, \mu_2, \nu_{12}),$

with relation $\quad \lambda_1 + \mu_2 + \nu_{12} = 0;$

W_3 (21): $\quad S = (\mu_1, \mu_2, \lambda_3, \lambda_4, \nu_{12}, \nu_{34}),$

with relation $\quad \mu_1 + \mu_2 + \lambda_3 + \lambda_4 + \nu_{12} + \nu_{34} = 0.$

In order to verify table 2 for ϕ_4, we have therefore to show that every face of R is equivalent to one of these three.

Now, following Voronoi's method, we consider the linear inequalities:

$$\left.\begin{aligned} L_1 &= 9p_{11} \geqslant 0, \\ M_1 &= 4p_{11} + \sum_2^6 p_{jj} - 4\sum_2^6 p_{1j} + 2\sum_{2\leqslant j<k\leqslant 6} p_{jk} \geqslant 0, \\ N_{12} &= 4p_{11} + 4p_{22} + \sum_3^6 p_{jj} + 8p_{12} - 4\sum_3^6 (p_{1j}+p_{2j}) + 2\sum_{3\leqslant j<k\leqslant 6} p_{jk} \geqslant 0, \end{aligned}\right\} \quad (8\cdot8)$$

corresponding to λ_1, μ_1, ν_{12} respectively, and those derived from them by all permutations of suffixes. A set $S = (\kappa_1, \ldots, \kappa_r)$ then determines a face of R when (*a*) equality in the remaining $27-r$ inequalities (8·8) determines the ratios of the p_{ij}; (*b*) this solution, with an appropriate sign, satisfies with strict inequalities the r relations (8·8) corresponding to $\kappa_1, \ldots, \kappa_r$.

The following identities are easily verified.

$$L_1+L_2+L_3+L_4+L_5+L_6 = M_1+M_2+M_3+M_4+M_5+M_6, \quad (8\cdot9)$$

$$L_1+M_1+N_{23}+N_{24}+N_{25}+N_{26} = L_2+M_2+N_{13}+N_{14}+N_{15}+N_{16}, \quad (8\cdot10)$$

$$L_1+L_2+L_3+N_{45}+N_{46}+N_{56} = M_4+M_5+M_6+N_{12}+N_{13}+N_{23}, \quad (8\cdot11)$$

These identities may be derived in the following way. The corresponding sets of six forms on each side of the identity, e.g. $(\lambda_1, \ldots, \lambda_6)$ and $(\mu_1, \ldots, \mu_6)$ in (8·9), have the properties: (*a*) no two forms in the same set combine; (*b*) each form in each set combines with all but one of the forms in the other set. The thirty-six distinct pairs of sets obtained in this way clearly correspond to the thirty-six double-sixes among the twenty-seven lines on a cubic surface, on translating 'forms' and 'combining' into 'lines' and 'intersecting'.

Using the identities, we establish

LEMMA 8·3. *Let S be a set of forms determining a face of R. Then if S contains λ_i, it contains a form from each of the sixteen sets*

$$(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6), \quad (8\cdot12)$$

$$(\lambda_j, \mu_j, \nu_{ik}, \nu_{il}, \nu_{ip}, \nu_{iq}), \quad (8\cdot13)$$

$$(\nu_{ij}, \nu_{ik}, \nu_{jk}, \mu_l, \mu_p, \mu_q), \quad (8\cdot14)$$

where i, j, k, l, p, q is any arrangement of 1, …, 6.

Proof. If S contains λ_1, then $L_1 > 0$. Since all expressions L, M, N are non-negative, we see that at least one expression on the right of each of (8·9), (8·10) and (8·11) is positive, and hence that S contains the corresponding form. The lemma now follows by applying all permutations of the suffixes.

We note that, by applying $\mathfrak{U}$, we obtain a similar result for forms containing some μ_i; we have simply to interchange λ and μ throughout.

We now establish systematically the three inequivalent sets $S = (\kappa_1, \ldots, \kappa_r)$ determining faces of R. Clearly $r \leqslant s-N+1 = 7$, and $r \geqslant 3$, since no two forms are dependent.

Since $\mathfrak{G}$ is transitive on the linear forms, we may take a form $S = (\lambda_1, \kappa_2, \ldots, \kappa_r)$. By lemma 8·3, S must now contain a form from each of the sixteen sets (8·12) to (8·14) with $i = 1$; and since $\kappa_2, \ldots, \kappa_r$ are at most six in number, some one of them, say κ_2, must occur in at least two of these sets. It follows that κ_2 must be one of $\mu_2, \ldots, \mu_6, \nu_{12}, \ldots, \nu_{16}$. By lemma 8·1 (ii), these ten forms are equivalent under $\mathfrak{G}(\lambda_1)$, and so S is equivalent to a set $(\lambda_1, \mu_2, \kappa_3, \ldots, \kappa_r)$.

Applying lemma 8·3 again with $i = 1$, we see that $\kappa_3, \ldots, \kappa_r$ must include a form from the eight sets (8·12) to (8·14) which do not contain μ_2, namely, (8·13) for $j = 3, 4, 5, 6$ and (8·14) for $j = 2$. Hence some one of them, say κ_3, occurs in at least two of these eight sets, and so must be one of ν_{12} (which occurs in all eight sets) or

$$\mu_3, \quad \mu_4, \quad \mu_5, \quad \mu_6, \quad \nu_{13}, \quad \nu_{14}, \quad \nu_{15}, \quad \nu_{16}. \tag{8·15}$$

If now $\kappa_3 = \nu_{12}$, the first linear relation (8·6) shows that $S = (\lambda_1, \mu_2, \nu_{12})$, determining the face W_2 (24).

We therefore exclude henceforth the set $(\lambda_1, \mu_2, \nu_{12})$ and all equivalent sets. Now κ_3 is one of the forms (8·15), and these are all equivalent under $\mathfrak{G}(\lambda_1, \mu_2)$ (which contains $\mathfrak{V}_{135}$ and all permutations of y_3, y_4, y_5, y_6). Hence S is equivalent to a set $(\lambda_1, \mu_2, \mu_3, \kappa_4, \ldots, \kappa_r)$; applying $\mathfrak{U}$ and a permutation of suffixes, we take S in the equivalent form

$$(\lambda_1, \lambda_2, \mu_3, \kappa_4, \ldots, \kappa_r).$$

Now S contains λ_1 and so also, by lemma 8·3, one of $\nu_{12}, \nu_{13}, \nu_{23}, \mu_4, \mu_5, \mu_6$. S cannot contain ν_{13} or ν_{23}, since the sets $(\lambda_1, \mu_3, \nu_{13})$ and $(\lambda_2, \mu_3, \nu_{23})$ are dependent. Hence S contains one of $\nu_{12}, \mu_4, \mu_5, \mu_6$, and these are equivalent under $\mathfrak{G}(\lambda_1, \lambda_2, \mu_3)$ (which contains $\mathfrak{V}_{124}$ and all permutations of y_4, y_5, y_6). Thus we may take $\kappa_4 = \mu_4$ and

$$S = (\lambda_1, \lambda_2, \mu_3, \mu_4, \kappa_5, \ldots, \kappa_r).$$

Reasoning as above, we see that S cannot now contain any of $\nu_{13}, \nu_{14}, \nu_{23}, \nu_{24}$. Applying lemma 8·3 with $i = 1$, we find just three sets which contain none of the known four forms in S; discarding ν_{13}, ν_{14} from them, we obtain

$$(\lambda_5, \mu_5, \nu_{12}, \nu_{16}), \quad (\lambda_6, \mu_6, \nu_{12}, \nu_{15}), \quad (\nu_{34}, \mu_2, \mu_5, \mu_6). \tag{8·16}$$

Since also λ_2 belongs to S, we obtain similarly (after discarding ν_{23} and ν_{24})

$$(\lambda_5, \mu_5, \nu_{12}, \nu_{26}), \quad (\lambda_6, \mu_6, \nu_{12}, \nu_{25}), \quad (\nu_{34}, \mu_1, \mu_5, \mu_6). \tag{8·17}$$

We now have at most three forms $\kappa_5, \ldots, \kappa_r$ which must include a form from each of the six sets (8·16) and (8·17). Hence κ_5, say, must occur in at least two of these sets, and so must be one of $\lambda_5, \mu_5, \lambda_6, \mu_6, \nu_{12}$. These five forms are equivalent under $\mathfrak{G}(\lambda_1, \lambda_2, \mu_3, \mu_4)$, and so we may take

$$S = (\lambda_1, \lambda_2, \mu_3, \mu_4, \mu_5, \kappa_6, \ldots, \kappa_r).$$

Of the sets (8·16) and (8·17), all are now accounted for except $(\lambda_6, \mu_6, \nu_{12}, \nu_{15})$, $(\lambda_6, \mu_6, \nu_{12}, \nu_{25})$. Since S cannot now contain ν_{15} or ν_{25}, it must therefore contain at least one of $\lambda_6, \mu_6, \nu_{12}$.

If now S contains λ_6, we obtain the dependent set $(\lambda_1, \lambda_2, \mu_3, \mu_4, \mu_5, \lambda_6)$, which is equivalent, by the transformation $\mathfrak{U}\mathfrak{V}_{346}$, to $(\mu_1, \mu_2, \lambda_3, \lambda_4, \nu_{12}, \nu_{34})$, determining the face W_3 (21). Otherwise S contains one of μ_6, ν_{12}, which are equivalent under $\mathfrak{V}_{126}$ (which leaves the five known forms of S invariant). Thus we may take

$$S = (\lambda_1, \lambda_2, \mu_3, \mu_4, \mu_5, \mu_6, \kappa_7).$$

(A further form κ_7 must appear, since the first six forms are independent.)

We conclude this analysis by showing that $\kappa_7 = \nu_{12}$. Since μ_6 belongs to S, lemma 8·3 shows that S contains one of the forms $\nu_{12}, \nu_{16}, \nu_{26}, \lambda_3, \lambda_4, \lambda_5$. But S cannot contain any of $\nu_{16}, \nu_{26}, \lambda_3, \lambda_4, \lambda_5$, in view of the dependent subsets $(\lambda_1, \mu_6, \nu_{16})$, $(\lambda_1, \lambda_2, \mu_4, \mu_5, \mu_6, \lambda_3)$ (and similar ones obtained by permuting suffixes). Thus S contains ν_{12} and so

$$S = (\lambda_1, \lambda_2, \mu_3, \mu_4, \mu_5, \mu_6, \nu_{12}) \sim (\mu_1, \mu_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \nu_{12}),$$

determining the face W_1 (20).

It remains for us to show that the neighbours of ϕ_4 along the faces W_1 (20), W_2 (24), W_3 (21) are equivalent respectively to ϕ_1, ϕ_2, ϕ_5. The explicit equations of these faces are easily found by the method of lemma 2·1 (the coefficients α_k having been established above). We obtain

$$\psi_1(\mathbf{x}) = -x_1x_2 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6,$$

$$\psi_2(\mathbf{x}) = 2x_1^2 + x_1x_2 + 2x_1x_3 + 2x_1x_4 + 2x_1x_5 + 2x_1x_6 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6,$$

$$\psi_3(\mathbf{x}) = x_5x_6.$$

Then we obtain the corresponding neighbours

$$f_1(\mathbf{x}) = \phi_4(\mathbf{x}) + \tfrac{1}{2}\psi_1(\mathbf{x}) = \phi_0(\mathbf{x}) - x_1x_2 = \phi_1(\mathbf{x});$$

$$f_2(\mathbf{x}) = \phi_4(\mathbf{x}) + \tfrac{1}{2}\psi_2(\mathbf{x}) = \phi_0(\mathbf{x}) + x_1(x_1 + x_3 + x_4 + x_5 + x_6) \sim \phi_0(\mathbf{x}) - x_1x_2 - x_1x_3 = \phi_2(\mathbf{x}),$$

under the transformation $x_3 \to -x_1 - x_2 - x_3 - x_4 - x_5 - x_6$,

$$f_3(\mathbf{x}) = \phi_4(\mathbf{x}) + \tfrac{1}{2}\psi_3(\mathbf{x})$$
$$= \phi_0(\mathbf{x}) - \tfrac{1}{2}(x_1x_2 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6);$$

but this is the form (4·13) (with $\rho = \frac{1}{2}$), which was shown in §4 to be equivalent to (4·9), i.e. to $\phi_5(\mathbf{x})$.

9. ϕ_2 AND ITS NEIGHBOURS

Following Voronoi, we define for $n = 6, 7, 8$

$$\phi_2(\mathbf{x}) = \phi_0(\mathbf{x}) - x_1x_2 - x_1x_3. \tag{9·1}$$

Making the transformation $\mathbf{x} = \mathfrak{T}\mathbf{y}$:

$$3x_1 = \sum_1^n y_k, \quad 3x_2 = \sum_1^n y_k - 3y_2, \quad 3x_3 = \sum_1^n y_k - 3y_3, \quad x_i = -y_i \quad (i = 4, \ldots, n), \tag{9·2}$$

of determinant $-\frac{1}{3}$, we obtain

$$18\phi_2 = 8\sum_1^n y_i^2 - 2\sum_{i<j} y_iy_j = (9-n)\sum_1^n y_i^2 + \sum_{i<j}(y_i - y_j)^2. \tag{9·3}$$

(9·2) and (9·3) give at once $\qquad D(\phi_2) = \dfrac{9-n}{2^n}.$

In (9·2), $\mathbf{x}$ is integral if and only if $\mathbf{y}$ is integral and satisfies

$$\sum_1^n y_i \equiv 0 \pmod 3. \tag{9·4}$$

From (9·3) and (9·4) we find easily that $M(\phi_2) = 1$ and that the minimal vectors are, in y co-ordinates,

$$(1, -1, 0, \ldots, 0)', \quad (1, 1, 1, 0, \ldots, 0)', \quad (1, 1, 1, 1, 1, 1, 0, \ldots, 0)', \tag{9·5}$$

$$(2, 1, 1, 1, 1, 1, 1, 1)' \tag{9·6}$$

(where the prime denotes all permutations of the co-ordinates, and the sets (9·6) exist only for $n = 8$). Thus

$$s = \binom{n}{2} + \binom{n}{3} + \binom{n}{6} + 8\binom{n}{8} \quad (n = 6, 7, 8)$$

(in agreement with Coxeter (1951, p. 420), where the form is denoted by E_n). It is easily verified from this result that ϕ_2 is perfect.

We now restrict ourselves to the case $n = 6$, $s = 36$. The associated linear forms, in variables contragredient to those of (9·3), are

$$\left.\begin{aligned} \lambda_{ij} &= y_i - y_j \qquad (1 \leqslant i < j \leqslant 6), \\ \mu_{ijk} &= y_i + y_j + y_k \quad (1 \leqslant i < j < k \leqslant 6), \\ \nu &= \sum_{i=1}^{6} y_i. \end{aligned}\right\} \tag{9·7}$$

The adjoint of (9·3), for $n = 6$, is a multiple of

$$\omega(\mathbf{y}) = 16\sum_1^6 y_i^2 + 8\sum_{i<j} y_i y_j = \Sigma\lambda_{ij}^2 + \Sigma\mu_{ijk}^2 + \nu^2,$$

so that ϕ_2 is eutactic, and hence extreme.

We now consider the group $\mathfrak{G}$. Arguing as before, we see that $\mathfrak{G}$ is the group of transformations of $x_1, \ldots, x_6$ which permute the associated linear forms (9·7). Here $\mathbf{x}$ and $\mathbf{y}$ are related by the transformation $\mathbf{y} = \mathfrak{T}'\mathbf{x}$ contragredient to (9·2). It is easily verified that the following transformations belong to $\mathfrak{G}$:

$$\mathfrak{P}\colon \text{ all permutations of } y_1, \ldots, y_6;$$

$$\mathfrak{R}\colon\ y_i \to \frac{1}{3}\sum_1^6 y_k - y_i \quad (i = 1, \ldots, 6).$$

$\mathfrak{P}$ leaves ν invariant and induces the corresponding permutation of suffixes in λ_{ij}, μ_{ijk}. $\mathfrak{R}$ leaves ν and each λ_{ij} invariant, and interchanges μ_{ijk} and μ_{lmn} (for any arrangement $i, \ldots, n$ of $1, \ldots, 6$).

We use the idea of combining forms introduced in § 8. All combining pairs are: (i) ν and μ_{ijk}; (ii) two μ's with two or no common suffixes; (iii) two λ's with one common suffix; (iv) a λ and a μ with one common suffix. The corresponding linear relations are (with suitable signs in each case)

$$\nu + \mu_{ijk} + \mu_{lmn} = 0, \quad \mu_{ijk} + \mu_{ijl} + \lambda_{kl} = 0, \quad \lambda_{ij} + \lambda_{ik} + \lambda_{jk} = 0.$$

LEMMA 9·1. (i) *$\mathfrak{G}$ is transitive on the associated linear forms.*

(ii) *$\mathfrak{G}(\nu)$ contains* $\mathfrak{P}$ and $\mathfrak{R}$ and has transitive systems (λ_{ij}), (μ_{ijk}).*

(iii) *Any set of four or five forms, every pair of which combines, is equivalent to $(\nu, \mu_{123}, \mu_{124}, \mu_{125})$ or $(\nu, \mu_{123}, \mu_{124}, \mu_{125}, \mu_{126})$ respectively.*

Proof. (ii) is clear from our above remarks. To prove (i), it therefore suffices to exhibit elements $\mathfrak{T}_1, \mathfrak{T}_2$ of $\mathfrak{G}$ with $\mathfrak{T}_1(\lambda_{12}) = \nu$, $\mathfrak{T}_2(\mu_{123}) = \nu$. We may in fact take

$$\mathfrak{T}_1\colon\ x_1 \to x_3, \quad x_2 \to 2x_1 + x_2 + x_3 - x_4 - x_5 - x_6, \quad x_3 \to x_1, \quad x_i \to x_1 + x_3 - x_i \quad (i = 4, 5, 6);$$

$$\mathfrak{T}_2\colon\ x_1 \to 2x_1 + x_2 + x_3 - x_4 - x_5 - x_6, \quad x_2 \to -x_2, \quad x_3 \to -x_3, \quad x_i \to x_1 - x_i \quad (i = 4, 5, 6);$$

since $\qquad \lambda_{12} = x_2, \quad \mu_{123} = x_1, \quad \nu = 2x_1 + x_2 + x_3 - x_4 - x_5 - x_6.$

* $\mathfrak{G}(\nu)$ is in fact generated by $\mathfrak{P}$, $\mathfrak{R}$ and $-\mathfrak{I}$.

For (iii), let $S = (\kappa_1, \kappa_2, \ldots, \kappa_r)$ $(r = 4$ or $5)$. By (i), S is equivalent to a set $(\nu, \kappa_2, \ldots, \kappa_r)$; and since all κ_i combine with ν, they are all μ's. By (ii), $S \sim (\nu, \mu_{123}, \kappa_3, \ldots, \kappa_r)$. Now κ_3 cannot be μ_{456}, since no other μ_{ijk} can combine with both μ_{123} and μ_{456}. Thus κ_3 must be one of μ_{12i}, μ_{13i}, μ_{23i} $(i = 4, 5, 6)$, and these are clearly equivalent under $\mathfrak{G}(\nu, \mu_{123})$. Hence $S \sim (\nu, \mu_{123}, \mu_{124}, \kappa_4, \ldots, \kappa_r)$.

Now κ_4 must be one of μ_{125}, μ_{126}, μ_{134}; also $(\nu, \mu_{123}, \mu_{124}, \mu_{125}) \sim (\nu, \mu_{123}, \mu_{124}, \mu_{126})$ under interchange of y_5, y_6, while $(\nu, \mu_{123}, \mu_{124}, \mu_{125}) \sim (\nu, \mu_{123}, \mu_{124}, \mu_{134})$, using $\mathfrak{R}$ and a suitable element of $\mathfrak{P}$. Hence, for $r = 4$, $S \sim (\nu, \mu_{123}, \mu_{124}, \mu_{125})$, as required. For $r = 5$,

$$S \sim (\nu, \mu_{123}, \mu_{124}, \mu_{125}, \kappa_5),$$

where clearly κ_5 can now only be μ_{126}. This proves the lemma.

We now proceed to the determination of the inequivalent faces of $R = R(\phi_2)$. As in § 8, we denote generally by S the set of forms lying off a face W of R, and so determining W; such a set we call a face-set.

The method of lemma 2·1 is not as useful here as with the previous forms, since the analysis of linear relations among thirty-six forms in twenty-one variables is very tedious. It was of practical use here in (*a*) obtaining a large number of faces by inspection; (*b*) checking quickly whether or not a given set of forms can be a subset of a face-set. The relevant relation

$$\sum_{i<j} \alpha_{ij} \lambda_{ij}^2 + \sum_{i<j<k} \beta_{ijk} \mu_{ijk}^2 + \gamma \nu^2 \equiv 0 \tag{9·8}$$

(identically in $y_1, \ldots, y_6$) yields the twenty-one equations*

$$\sum_j \alpha_{ij} + \sum_{j,k} \beta_{ijk} + \gamma = 0 \quad (i = 1, \ldots, 6), \tag{9·9}$$

$$-\alpha_{ij} + \sum_k \beta_{ijk} + \gamma = 0 \quad (i, j = 1, \ldots, 6) \tag{9·10}$$

(where, for convenience, we disregard the order of the suffixes). Summing (9·10) over j and using (9·9) we obtain

$$\sum_{j,k} \beta_{ijk} + 2\gamma = 0, \tag{9·11}$$

$$\sum_j \alpha_{ij} - \gamma = 0. \tag{9·12}$$

From (9·12) we deduce the useful result:

LEMMA 9·2. *No set* $(\kappa_1, \kappa_2, \ldots, \kappa_6)$ *can be a subset of a face-set S if it has the properties*: (*a*) κ_1 *combines with none of* $\kappa_2, \ldots, \kappa_6$; (*b*) *all pairs from* $\kappa_2, \ldots, \kappa_6$ *combine.*

Proof. By lemma 2·1 and (9·12), S cannot have a subset $(\nu, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16})$. Now, by lemma 9·1 (iii), all sets $(\kappa_2, \ldots, \kappa_6)$ satisfying (*b*) are equivalent, and hence are equivalent to $(\lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16})$; and under this equivalence relation, we must have $\kappa_1 \sim \nu$, since ν is the unique form not combining with any of $\lambda_{12}, \ldots, \lambda_{16}$. The lemma follows at once.

In the main we shall use Voronoi's method, along precisely the same lines as in § 8.

* Professor Room suggested that a complete solution of (9·9) and (9·10) could be found from the table of synthemes used by Baker in his discussion of the Pascal figure [*Principles of geometry*, vol. 2, p. 221] (see appendix).

We have the thirty-six inequalities

$$\left.\begin{aligned} L_{ij} &= p_{ii}+p_{jj}-2p_{ij} \geqslant 0 \quad (1 \leqslant i < j \leqslant 6), \\ M_{ijk} &= p_{ii}+p_{jj}+p_{kk}+2p_{ij}+2p_{ik}+2p_{jk} \geqslant 0 \quad (1 \leqslant i < j < k \leqslant 6), \\ N &= \sum_1^6 p_{ii}+2\sum_{i<j} p_{ij} \geqslant 0, \end{aligned}\right\} \tag{9·13}$$

arising from λ_{ij}, μ_{ijk}, ν respectively. We simplify the algebra by setting

$$u_i = 3p_{ii} \quad (i = 1, \ldots, 6), \qquad v_{ij} = p_{ii}+p_{jj}-2p_{ij} \;\; (= v_{ji}) \quad (i \neq j), \tag{9·14}$$

whence

$$\left.\begin{aligned} L_{ij} &= v_{ij}, \\ M_{ijk} &= u_i+u_j+u_k-v_{ij}-v_{ik}-v_{jk}, \\ N &= 2\Sigma u_i - \sum_{i<j} v_{ij}. \end{aligned}\right\} \tag{9·15}$$

We find the following identities (which are easily built up, as in § 8, by considering combinations among the corresponding forms):

$$N+L_{12}+L_{34}+L_{56} = M_{135}+M_{146}+M_{236}+M_{245},$$
$$L_{12}+L_{34}+M_{125}+M_{345} = L_{13}+L_{24}+M_{135}+M_{245},$$
$$N+M_{123}+L_{12}+L_{13}+L_{45}+L_{46} = L_{14}+M_{156}+M_{124}+M_{134}+M_{236}+M_{235}.$$

From these, and those derived from them by permuting suffixes, we derive (as in § 8):

Lemma 9·3. *Let S be any face-set. Then*

(i) *If $\nu \in S$, S contains a form from every set*

$$(\mu_{ikm}, \mu_{iln}, \mu_{jkn}, \mu_{jlm}). \tag{9·16}$$

(ii) *If $\nu \in S$, S contains a form from every set*

$$(\lambda_{ij}, \mu_{ijk}, \mu_{ijl}, \mu_{klm}, \mu_{kln}, \mu_{imn}). \tag{9·17}$$

(iii) *If $\mu_{ijk} \in S$, S contains a form from every set*

$$(\lambda_{il}, \lambda_{jm}, \mu_{ilk}, \mu_{jmk}). \tag{9·18}$$

(iv) *If $\lambda_{ij} \in S$, S contains a form from every set*

$$(\lambda_{ik}, \lambda_{jl}, \mu_{ikm}, \mu_{jlm}). \tag{9·19}$$

In each case, the suffixes $i, j, \ldots$ are supposed distinct but otherwise arbitrary.

For convenience of printing, the triple ijk will be written for μ_{ijk}. It will also be convenient to interpret the subgroup $\mathfrak{P}$ of $\mathfrak{G}$ as the permutation group on the suffixes $1, 2, \ldots, 6$ and to write elements of $\mathfrak{P}$ in cycle notation; we shall further denote the permutation group on $i, j, \ldots, k$ by $(i, j, \ldots, k)'$.

We now examine the implications of lemma 9·3 (i), and prove

Lemma 9·4. *Any set of forms μ_{ijk} which contains a form from each of the sets* (9·16) *has a subset equivalent, under the group* $\{\mathfrak{P}, \mathfrak{R}\}$, *to one of*

$$T_1 = (123, 124, 135, 145, 236, 246, 356, 456),$$
$$T_2 = (123, 124, 125, 134, 136, 245, 256, 346, 356, 456),$$
$$T_3 = (123, 124, 125, 126, 134, 135, 246, 256),$$
$$T_4 = (123, 124, 125, 126, 134, 135, 234, 246),$$
$$T_5 = (123, 124, 125, 126, 134, 135, 136),$$
$$T_6 = (123, 124, 125, 134, 135, 145).$$

Proof. From any such set, we can choose a subset T which satisfies the condition minimally, i.e. contains no subset satisfying the condition of containing a form from each set (9·16). We prove the lemma by showing that any 'minimal' set T is equivalent under $\{\mathfrak{P}, \mathfrak{R}\}$ to one of $T_1, \ldots, T_6$ (which are easily verified to be minimal).

(i) We first show that T has a subset equivalent to (123, 124). By lemma 9·1 (ii), $T \sim (123, \ldots)$. T contains one of 124, 135, 236, 456, and clearly $124 \sim 135 \sim 236$ under $\{(1,2,3)', (4,5,6)'\}$. Hence either $T \sim (123, 124, \ldots)$, as required, or $T \sim (123, 456, \ldots)$. In the latter case, any further triple has two suffixes in common with either 123 or 456, and so once again T has a subset equivalent under $\mathfrak{P}$ to (123, 124).

Passing to an equivalent set, we may now take

$$T = (123, 124, \ldots).$$

(ii) Suppose now that T has no subset $\sim (123, 124, 125)$; we show that then $T \sim T_1$. By lemma 9·1 (iii) (since $\{\mathfrak{P}, \mathfrak{R}\}$ leaves ν invariant) T can contain no three forms every pair of which combines. Thus now

$$125, 126, 134, 234 \notin T.$$

T contains one of 135, 146, 236, 245, and these are equivalent under $\{(12), (34), (56)\}$, which leaves (123, 124) invariant; hence we may take $T = (123, 124, 135, \ldots)$. Now

$$136, 235 \notin T.$$

T contains one of 125, 136, 234, 456, and so contains 456:

$$T = (123, 124, 135, 456, \ldots).$$

T contains one of 125, 134, 246, 356, and so one of 246, 356; since $246 \sim 356$ under (23) (45), we may take

$$T = (123, 124, 135, 456, 356, \ldots).$$

Now
$$156, 256, 345, 346 \notin T.$$

T contains one of 125, 136, 246, 345, and hence 246; it also contains one of 126, 145, 235, 346, and hence 145. Thus now

$$T = (123, 124, 135, 456, 356, 246, 145, \ldots).$$

Finally, T contains one of 134, 156, 236, 245, from which 236 is now the only possible choice. We therefore have $T = T_1$.

We need now consider only sets equivalent to

$$T = (123, 124, 125, \ldots). \tag{9·20}$$

(iii) Suppose next that T, given by (9·20), contains no subset equivalent to (123, 124, 125, 126); we show that then $T \sim T_2$ or $T \sim T_6$. We note first that the known subset (123, 124, 125) of T is invariant under the group $\{(12), (3,4,5)'\}$.

(*a*) As a first step, we show that T is equivalent to a set

$$(123, 124, 125, 134, \ldots).$$

For, if not, certainly $134 \notin T$; using the above group, we see that therefore

$$134, 135, 145, 235, 235, 245 \notin T.$$

Now T contains one of 126, 134, 235, 456; and $126 \notin T$, by our initial assumption. Hence $456 \in T$; from the group $\{(12), (3, 4, 5)\}'$, we have also $356 \in T$, $346 \in T$. Thus now

$$T = (123, 124, 125, 456, 356, 346, \ldots).$$

It is now easily verified that no further triple can belong to T, in particular, none of 134, 156, 236, 245; and this is impossible.

Thus now we may take $\qquad T = (123, 124, 125, 134, \ldots).$ $\qquad$ (9·21)

(*b*) Suppose now that $135 \in T$, so that

$$T = (123, 124, 125, 134, 135, \ldots).$$

By our original assumption, $\qquad 126, 234, 136, 235 \notin T$

(each of these giving a subset $\sim (123, 124, 125, 126)$).

All sets (9·16) are now accounted for with the exception of (126, 145, 234, 356), (126, 145, 235, 346), (136, 145, 235, 246), (136, 145, 234, 256). If now $145 \in T$, we have $T = T_6$. If, however, $145 \notin T$, we see that T contains all of 356, 346, 246, 256:

$$T = (123, 124, 125, 134, 135, 356, 346, 246, 256, \ldots).$$

But now 123 is redundant (since every set (9·16) containing 123 also contains one of 346, 356, 246, 256); thus T cannot be minimal.

(*c*) We have, finally, to consider here sets (9·21) not containing 135 (or, by our original assumption, 126). Since the known subset (123, 124, 125, 134) is invariant under $\{(34), \mathfrak{R}(16)(25)\}$, we therefore have

$$126, 234, 135, 145 \notin T.$$

Since T contains a form from each set (126, 135, 234, 456), (126, 145, 234, 356), we have at once $456 \in T$, $356 \in T$,

$$T = (123, 124, 125, 134, 456, 356, \ldots).$$

T contains a form from each set (126, 135, 245, 346), (126, 145, 235, 346), and hence one from each pair (245, 346), (235, 346). If both 235, $245 \in T$, 134 is redundant and T is not minimal. Hence $346 \in T$. Applying the group $\{(34), \mathfrak{R}(16)(25)\}$, we see that also $256 \in T$, and so now

$$T = (123, 124, 125, 134, 456, 356, 346, 256, \ldots).$$

The set of eight known forms of T is invariant under $\{(34), \mathfrak{R}, (16)(25)\}$.

Of the sets (9·21), (135, 146, 236, 245) and (136, 145, 235, 246) are still not accounted for. As above, T cannot contain 135 or 145; and, applying the group $\{(34), \mathfrak{R}, (16)(25)\}$, we see that T cannot contain 246 or 236. Thus T contains a form from each pair (146, 245), (136, 235). Since $136 \sim 235$ under $\mathfrak{R}(34)$, we may suppose that $136 \in T$. If now $146 \in T$, it is easily verified that 125 is redundant (every set (9·21) which contains 125 also contains one of 456, 356, 136, 146). Hence finally $245 \in T$ and we have $T = T_2$.

(iv) These results show that we may henceforth confine ourselves to sets of the form

$$T = (123, 124, 125, 126, \ldots).$$

Now T contains one of 134, 156, 235, 246, and these forms are equivalent under $\{(12), (3,4,5,6)'\}$; hence we may take

$$T = (123, 124, 125, 126, 134, \ldots). \qquad (9\cdot22)$$

(*a*) Suppose first that $135 \in T$,

$$T = (123, 124, 125, 126, 134, 135, \ldots).$$

T contains a form from each of the sets (136, 145, 234, 256), (136, 145, 235, 246). Clearly $145 \notin T$, since otherwise T has T_6 as a proper subset and so is not minimal. Also, if $136 \in T$, we have at once $T = T_5$. There remains only to consider the case when T contains a form from each pair (234, 256), (235, 246).

Now T cannot contain both 234 and 235, since otherwise it has the proper subset (123, 124, 125, 134, 135, 234, 235), which is equivalent to T_5 under $\mathfrak{R}(16)\,(25)\,(34)$. Hence T must contain one of 246, 256; and these are equivalent under (45), which leaves the known subset (123, 124, 125, 126, 134, 135) of T invariant. Hence we may take

$$T = (123, 124, 125, 126, 134, 135, 246, \ldots).$$

Now T contains one of 234, 256, from which we have at once $T = T_4$ or $T = T_3$ respectively.

(*b*) Suppose, finally, that T, given by (9·22), does not contain 135. Since the known subset (123, 124, 125, 126, 134) of T is invariant under $\{(34), (56)\}$, we may suppose that

$$135, 136, 145, 136 \notin T. \qquad (9\cdot23)$$

T contains one of 136, 145, 235, 246, and hence one of 235, 246. Since $235 \sim 246$ under $\{(34), (56)\}$, we may take

$$T = (123, 124, 125, 126, 134, 235, \ldots).$$

The set (123, 124, 125, 126, 134, 235) is invariant under (12) (45); hence, from (9·23), we see that also

$$234, 236, 245, 256 \notin T.$$

But now T cannot contain any form from the set (136, 145, 234, 256), which is impossible.

This completes the proof of the lemma.

Our object is now to specify, as far as possible, a unique representative of each class of equivalent faces of R. Such a specification is difficult, but the following result goes a long way in this direction:

LEMMA 9·5. *Every face-set is equivalent to a set S with the following properties:*

(i) *S contains ν;*

(ii) *S contains p forms λ_{ij} and q forms μ_{ijk}, where*

$$p + q \leqslant 15;$$

(iii) *The number of forms of S which do not combine with any given form of S is at most p.*

Proof. (i) is clear, since $\mathfrak{G}$ is transitive on the associated linear forms. (ii) is true whenever (i) is, since S can contain at most $s - N + 1 = 16$ forms.

Finally, we observe that if S is transformed so that any particular form, κ, is transformed into ν, then a form not combining with κ is transformed into a λ_{ij}, and one combining with

κ into a μ_{ijk}. Hence property (iii) may always be achieved by transforming into ν that form of S which fails to combine with a maximum number of forms of S.

Henceforward, then, we need *consider only face-sets S satisfying the condition of lemma* 9·5.

Since $\nu \in S$, lemmas 9·3 (i) and 9·4 show that, after applying a suitable transformation of $\mathfrak{G}(\nu)$, S has a subset (ν, T_i) for some $i = 1, 2, \ldots, 6$. Since these six cases are not mutually exclusive, we eliminate repetition by considering each case: $(\nu, T_i) \subset S$, *under the assumption that S has no subset equivalent under* $G(\nu)$ *to* (ν, T_j) *for any* $j<i$.

We shall show, in the following six lemmas, that all face-sets S satisfying these two conditions are equivalent under $\mathfrak{G}(\nu)$ to one of the following sixteen sets (where we have included in brackets the number of forms in each set):

$$S_1(11) = (\nu, \lambda_{14}, \lambda_{15}, \lambda_{16}, 123, 124, 125, 126, 134, 135, 136),$$
$$S_2(16) = (\nu, \lambda_{15}, \lambda_{16}, \lambda_{23}, \lambda_{26}, \lambda_{35}, 123, 124, 125, 134, 136, 245, 256, 346, 356, 456),$$
$$S_3(16) = (\nu, \lambda_{14}, \lambda_{15}, \lambda_{23}, \lambda_{24}, \lambda_{25}, \lambda_{34}, \lambda_{35}, 123, 124, 125, 126, 134, 135, 136, 236),$$
$$S_4(8) = (\nu, \lambda_{16}, 123, 124, 125, 134, 135, 145),$$
$$S_5(12) = (\nu, \lambda_{23}, \lambda_{24}, \lambda_{25}, 123, 124, 125, 134, 135, 145, 126, 345),$$
$$S_6(14) = (\nu, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{25}, \lambda_{45}, 123, 124, 125, 126, 134, 135, 234, 246),$$
$$S_7(16) = (\nu, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{23}, \lambda_{24}, \lambda_{25}, \lambda_{45}, 123, 124, 125, 126, 134, 135, 246, 256),$$
$$S_8(12) = (\nu, \lambda_{16}, \lambda_{25}, \lambda_{34}, 123, 124, 135, 145, 236, 246, 356, 456),$$
$$S_9(16) = (\nu, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{24}, \lambda_{25}, \lambda_{45}, 123, 124, 125, 126, 134, 135, 246, 256, 456),$$
$$S_{10}(15) = (\nu, \lambda_{23}, \lambda_{24}, \lambda_{25}, \lambda_{34}, \lambda_{35}, 123, 124, 125, 126, 134, 135, 136, 145, 236),$$
$$S_{11}(16) = (\nu, \lambda_{14}, \lambda_{16}, \lambda_{23}, \lambda_{25}, \lambda_{34}, \lambda_{36}, \lambda_{45}, 123, 124, 125, 126, 134, 135, 234, 246),$$
$$S_2'(16) = (\nu, \lambda_{14}, \lambda_{16}, \lambda_{36}, \lambda_{45}, \lambda_{46}, 123, 124, 125, 126, 134, 135, 246, 256, 156, 234),$$
$$S_2''(16) = (\nu, \lambda_{14}, \lambda_{23}, \lambda_{24}, \lambda_{34}, \lambda_{46}, 123, 124, 125, 126, 134, 135, 136, 145, 245, 345),$$
$$S_5'(12) = (\nu, \lambda_{16}, \lambda_{23}, \lambda_{36}, 123, 124, 125, 126, 134, 135, 246, 256),$$
$$S_6'(14) = (\nu, \lambda_{14}, \lambda_{15}, \lambda_{23}, \lambda_{34}, \lambda_{35}, 123, 124, 125, 126, 134, 135, 136, 245),$$
$$S_{10}'(15) = (\nu, \lambda_{14}, \lambda_{16}, \lambda_{23}, \lambda_{24}, \lambda_{45}, 123, 124, 125, 126, 134, 135, 246, 256, 346).$$

The first eleven face-sets $S_1, \ldots, S_{11}$ are in fact inequivalent under the full group $\mathfrak{G}$; and we shall later show (lemma 9·13) that the last five are equivalent to the corresponding unprimed sets.

LEMMA 9·6. *If S contains* (ν, T_1), *then*

$$S \sim S_8 = (\nu, \lambda_{16}, \lambda_{25}, \lambda_{34}, T_1).$$

Proof. We have

$$(\nu, T_1) = (\nu, 123, 124, 135, 145, 236, 246, 356, 456),$$

and $\mathfrak{G}(\nu, T_1)$ contains $\mathfrak{H} = \{\mathfrak{R}, (16), (25), (34), (12)(56), (13)(46), (23)(45)\}$, under which all remaining twelve μ_{ijk} are equivalent.

(i) Suppose first that S contain no further μ_{ijk}. By lemma 9·3 (ii), S contains a form from the set $(\lambda_{16}, 126, 156, 235, 245, 134)$, so that S contains λ_{16}. Applying $\mathfrak{H}$, we see that S contains also λ_{25} and λ_{34}. Thus $S = S_8$, as asserted.

Henceforward, then, we may assume that S contains some further μ_{ijk}, and therefore that $125 \in S$,

$$(\nu, T_1, 125) \subset S.$$

We note that $\mathfrak{G}(\nu, T_1, 125)$ contains $\mathfrak{H}' = \{(25), (34), \mathfrak{R}(16)(23)(45)\}$, under which

$$126 \sim 156 \sim 245 \sim 235.$$

It is easy to show that $134 \notin S.$

For, if $134 \in S$, then 134 fails to combine with 125, 236, 246, 356, 456, so that

$$p \geqslant 5, \quad q \leqslant 15 - p \leqslant 10.$$

Since S already contains ten μ_{ijk}, we require $q = 10$, $p = 5$, and S can contain no further μ_{ijk}. Now S contains one of $(\lambda_{16}, 345, 234, 146, 136, 256)$ and so $\lambda_{16} \in S$. Thus S has as a proper subset the face-set $S_4 = (\nu, \lambda_{16}, T_6)$, which is impossible.

(ii) Suppose now that also $346 \in S$, so that

$$(\nu, T_1, 125, 346) \subset S.$$

$\mathfrak{G}(\nu, T_1, 125, 346)$ contains $\{\mathfrak{R}, (25), (34), (16)(23)(45)\}$, under which all of 126, 136, 146, 156, 234, 235, 245, 345 are equivalent, and $134 \sim 256$. If now S contains any further μ, we may suppose it to be either 126 or 134. As above, $134 \notin S$. Also, if $126 \in S$, we have $q \geqslant 11$, whence $p \leqslant 4$; but 126 fails to combine with 135, 145, 346, 356, 456, whence $p \geqslant 5$. This contradiction shows that S contains no further μ.

Since S contains one of $(\lambda_{16}, 126, 156, 235, 245, 134)$, we have

$$\lambda_{16} \in S.$$

Since $123 \in S$, lemma 9·3 (iii) shows that S contains one of λ_{14}, λ_{25}, 134, 235, and hence one of λ_{14}, λ_{25}. Applying the above group, we see that S contains a form from each pair

$$(\lambda_{14}, \lambda_{25}), \quad (\lambda_{13}, \lambda_{25}), \quad (\lambda_{56}, \lambda_{34}), \quad (\lambda_{26}, \lambda_{34}).$$

Now clearly not both λ_{25}, λ_{34} belong to S, else S has S_8 as a proper subset. Since $\lambda_{25} \sim \lambda_{34}$ under (16) (23) (45), we may suppose that $\lambda_{25} \notin S$; then S must contain both λ_{14}, λ_{13} and we have

$$(\nu, \lambda_{16}, \lambda_{13}, \lambda_{14}, T_1, 125, 346) \subset S.$$

We now show that this is impossible.

In the given subset of S, each of the ten forms λ_{13}, λ_{14}, 123, 124, 135, 145, 236, 246, 356, 456 fails to combine with five other forms; hence $p \geqslant 5$, and, since $q = 10$, we must have $p = 5$. This means that any further form of S must combine with all ten forms above. The only such form is λ_{34}. But this means that S contains at most four forms λ_{ij}, $p \leqslant 4$; a contradiction.

(iii) Suppose next that $346 \notin S$. We note that all our conditions are still invariant under $\mathfrak{H}'$. Now S contains a form from each of the sets

$$(\lambda_{16}, 126, 156, 235, 245, 134), \quad (\lambda_{25}, 235, 245, 134, 346, 126), \quad (\lambda_{34}, 134, 346, 126, 156, 235).$$

Since $346 \notin S$, by hypothesis, and $134 \notin S$, it follows that *either* S contains one of 126, 156, 235, 245 *or* S contains all of λ_{16}, λ_{25}, λ_{34}. The latter alternative is impossible, from the subset S_8.

Hence S contains one of 126, 156, 235, 245; since these are all equivalent under $\mathfrak{H}'$, we may suppose that

$$(\nu, T_1, 125, 126) \subset S.$$

Now $q \geqslant 10$; also $p \geqslant 5$, since both 356, 456 fail to combine with five other forms of S. Hence $p = 5$, $q = 10$, and S contains no further μ_{ijk}. Also since $p = 5$, all further forms in S (which must be five λ_{ij}) must combine with both 356 and 456; in particular, S cannot contain any of λ_{13}, λ_{14}, λ_{56}.

Now $123 \in S$ and so, by Lemma 9·3 (iii), S contains one of $(\lambda_{14}, \lambda_{25}, 134, 235)$; hence $\lambda_{25} \in S$. Similarly, since $135 \in S$, S contains one of $(\lambda_{34}, \lambda_{56}, 134, 156)$; hence $\lambda_{34} \in S$. But now 125 fails to combine with six forms of S, viz. $\lambda_{25}, \lambda_{34}, 236, 246, 356, 456$, contradicting $p = 5$.

This contradiction shows that this case is impossible, and the proof of the lemma is complete.

Lemma 9·7. *If S contains (ν, T_2), then*

$$S = S_2 = (\nu, \lambda_{15}, \lambda_{16}, \lambda_{23}, \lambda_{26}, \lambda_{35}, T_2)$$

Proof. The given subset

$$(\nu, T_2) = (\nu, 123, 124, 125, 134, 136, 245, 256, 346, 356, 456)$$

is invariant under the subgroup $\mathfrak{H} = \{\mathfrak{R}, (12)(35), (13)(26), (23)(56)\}$ of $\mathfrak{G}(\nu)$, under which all μ_{ijk} in the subset are equivalent. Since 123 fails to combine with the four forms 245, 256, 346, 356, it follows that (i) every μ_{ijk} in (ν, T_2) fails to combine with four forms of the subset; and (ii) $p \geqslant 4$.

We now show that S contains no further μ_{ijk}. For if it does, then certainly $q \geqslant 11$. Also, the added μ_{ijk} cannot combine with both 123 and 456, whence $p \geqslant 5$ and so $q \leqslant 10$; a contradiction.

Since $123 \in S$, lemma 9·3 (iii) shows that S contains one of $\lambda_{15}, \lambda_{24}, 135, 234$, and so one of λ_{15}, λ_{24}. Continuing this argument, and using the group $\mathfrak{H}$, we find that S must contain a form from every one of the twenty pairs λ_{ij}, λ_{kl} with distinct suffixes chosen from

$$\lambda_{14},\ \lambda_{24},\ \lambda_{34},\ \lambda_{45},\ \lambda_{46};\quad \lambda_{15},\ \lambda_{16},\ \lambda_{23},\ \lambda_{26},\ \lambda_{35}.$$

Now S cannot contain all five λ_{i4} $(i = 1, \ldots, 6)$, since, by lemma 9·2, it cannot have a subset $(\nu, \lambda_{14}, \ldots, \lambda_{45}, \lambda_{46})$. Since all λ_{i4} are equivalent under $\mathfrak{H}$, we may suppose that, say, $\lambda_{14} \notin S$. As was just shown, S contains a form from each pair $(\lambda_{14}, \lambda_{23})$, $(\lambda_{14}, \lambda_{26})$, $(\lambda_{14}, \lambda_{35})$, and so S contains λ_{23}, λ_{26}, λ_{35}.

If now $\lambda_{15} \notin S$, S must contains all of λ_{24}, λ_{34}, λ_{46}, which is impossible, since it would give $p \geqslant 6$, $p+q \geqslant 16$. Hence $\lambda_{15} \in S$; similarly, $\lambda_{16} \in S$. This now gives at once $S = S_2$, as asserted.

Lemma 9·8. *If S contains (ν, T_3) (but no subset equivalent under $\mathfrak{G}(\nu)$ to (ν, T_1) or (ν, T_2)) then S is equivalent to one of*

$$\begin{aligned}
S_5' &= (\nu, \lambda_{16}, \lambda_{23}, \lambda_{36}, T_3),\\
S_7 &= (\nu, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{23}, \lambda_{24}, \lambda_{25}, \lambda_{45}, T_3),\\
S_{10}' &= (\nu, \lambda_{14}, \lambda_{16}, \lambda_{23}, \lambda_{24}, \lambda_{45}, T_3, 346),\\
S_9 &= (\nu, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{24}, \lambda_{25}, \lambda_{45}, T_3, 456),\\
S_2' &= (\nu, \lambda_{14}, \lambda_{16}, \lambda_{36}, \lambda_{45}, \lambda_{46}, T_3, 234, 156).
\end{aligned}$$

Proof. We have

$$(\nu, T_3) = (\nu, 123, 124, 125, 126, 134, 135, 246, 256)$$

and $\mathfrak{G}(\nu, T_3)$ contains the group $\mathfrak{H} = \{(45), (12)(36)\}$.

(*a*) Suppose first that
$$\lambda_{16}, \lambda_{23} \in S.$$

Since $\lambda_{16} \in S$, $145 \notin S$ (else S has the face-set S_4 as a proper subset); applying $\mathfrak{H}$, we see that also $245 \notin S$. Thus
$$145 \notin S, \quad 245 \notin S.$$

If now $\lambda_{36} \in S$, we have at once $S = S_5'$, as required. We may therefore assume that
$$\lambda_{36} \notin S.$$

Since $134 \in S$, lemma 9·3 (iii) shows that S contains a form from each set $(\lambda_{15}, \lambda_{36}, 145, 346)$, $(\lambda_{36}, \lambda_{45}, 136, 145)$, and so one of $(\lambda_{15}, 346)$ and one of $(\lambda_{45}, 136)$. Applying $\mathfrak{H}$ we see that S contains a form from each pair:
$$(\lambda_{15}, 346), \quad (\lambda_{25}, 346), \quad (\lambda_{14}, 356), \quad (\lambda_{24}, 356), \quad (\lambda_{45}, 136), \quad (\lambda_{45}, 236). \qquad (9·24)$$

If now $\lambda_{45} \notin S$, S must contain both 136 and 236, and this is easily seen to be impossible. For then we have $q \geqslant 10$ and so $p \leqslant 5$; also each of 136, 236 fails to combine with five forms of (ν, T_3), so that $p \geqslant 5$. Hence $p = 5$, $q = 10$ and S contains no further μ_{ijk}. But then, from the pairs (9·24), it follows that S contains all of $\lambda_{15}, \lambda_{25}, \lambda_{14}, \lambda_{24}$; thus now 136 fails to combine also with $\lambda_{25}, \lambda_{24}$, giving $p \geqslant 7$; a contradiction.

Thus $\lambda_{45} \in S$. From the remaining four pairs (9·24), we find three inequivalent possibilities:

(i) S contains both 346, 356; this must be discarded, since then S contains a subset $\sim (\nu, T_1)$ under (35).

(ii) S contains just one of 346, 356. Since $346 \sim 356$ under $\mathfrak{H}$, we may suppose that $346 \in S$, $356 \notin S$. From the pairs (9·24), we see that S contains λ_{14} and λ_{24}, so that $S = S_{10}'$, as required.

(iii) S contains neither of 346, 356. Then S must contain all of $\lambda_{15}, \lambda_{25}, \lambda_{14}, \lambda_{24}$, giving $S = S_7$.

(*b*) Suppose next that
$$\lambda_{16} \notin S, \quad \lambda_{23} \notin S.$$

By lemma 9·3 (ii), S contains a form from each set
$$(\lambda_{16}, 146, 156, 245, 345, 236), \quad (\lambda_{23}, 234, 235, 145, 456, 136).$$

Also, since $124 \in S$, S contains one of
$$(\lambda_{16}, \lambda_{23}, 146, 234);$$
and, since $125 \in S$, one of
$$(\lambda_{16}, \lambda_{23}, 156, 235).$$

Thus now S contains a form from each of the sets
$$(146, 156, 245, 345, 236), \quad (234, 235, 145, 456, 136), \quad (146, 234), \quad (156, 235). \qquad (9·25)$$

Now $146 \sim 234$ under $\mathfrak{H}$, so we may suppose that
$$234 \in S.$$

We show that it is now impossible that $235 \in S$. For, in this case, we should have
$$(\nu, 123, 124, 125, 126, 134, 135, 246, 256, 234, 235) \subset S,$$
where each of 135, 246 fails to combine with four other forms of the subset. S must still contain a form from the first set (9·25), so that $q \geqslant 11$ and $p \leqslant 4$. It follows that $q = 11, p = 4$,

and all further forms of S must combine with both 135, 246; no μ_{ijk} satisfies this condition, in particular none of (146, 156, 245, 345, 236).

Thus $235 \notin S$ and so, from the last pair (9·25), $156 \in S$. Thus now

$$(\nu, 123, 124, 125, 126, 134, 135, 246, 256, 234, 156) \subset S,$$

and $q \geq 10$, $p \leq 5$. We see that S can obtain no further μ_{ijk} (since this would give $p \geq 5$, $q \geq 11$, precisely as above).

Since $126 \in S$, S contains one of $(\lambda_{14}, \lambda_{23}, 146, 236)$ and hence λ_{14}; applying the transformation (12) (36) (45) (which leaves the given subset of S invariant), we see that also $\lambda_{25} \in S$. But now 256 fails to combine with the six forms $\lambda_{14}, \lambda_{25}$, 123, 124, 125, 234 of S, giving $p \geq 6$; a contradiction.

(*c*) Suppose finally that S contains just one of the forms $\lambda_{16}, \lambda_{23}$. Since $\lambda_{16} \sim \lambda_{23}$ under $\mathfrak{H}$, we may suppose that

$$\lambda_{16} \in S, \quad \lambda_{23} \notin S.$$

Since $\lambda_{16} \in S$, we have, as in (*a*), $\quad 145 \notin S.$

As in (*b*), S must contain one of $(\lambda_{23}, 234, 235, 145, 456, 136)$ and so now one of

$$(234, 235, 456, 136). \tag{9·26}$$

(i) Suppose $\quad 234 \notin S, \quad 235 \notin S.$

Now since $124 \in S$, S contains one of $(\lambda_{15}, \lambda_{23}, 145, 234)$ and hence λ_{15}; applying (45) (which leaves all our conditions invariant), we have also $\lambda_{14} \in S$.

Now S contains $(\nu, \lambda_{14}, \lambda_{15}, \lambda_{16}, 123, 124, 125, 126, 134, 135)$, and so $136 \notin S$ from the face-set S_1. From the set (9·26) we see that therefore $456 \in S$, and now

$$(\nu, \lambda_{14}, \lambda_{15}, \lambda_{16}, 123, 124, 125, 126, 134, 135, 246, 256, 456) \subset S. \tag{9·27}$$

It is easy to see that S can contain no further μ_{ijk}. For if it did we should have $q \geq 10$ and so $p \leq 5$; since each of 134, 135, 456 fails to combine with five forms of the subset (9·27), we should require $p = 5$ and therefore that the added μ_{ijk} combine with all of 134, 135, 456. The only triple fulfilling this requirement is 345; this fails to combine with six forms of the subset (9·27), and its presence is therefore incompatible with $p = 5$.

Since 134, 246, 256 belong to S, lemma 9·3 shows that S contains a form from each of the sets $(\lambda_{45}, \lambda_{36}, 136, 145)$, $(\lambda_{25}, \lambda_{36}, 245, 346)$, $(\lambda_{24}, \lambda_{36}, 245, 356)$. Since S contains none of the triples occurring here, it must contain a λ_{ij} from each pair $(\lambda_{45}, \lambda_{36})$, $(\lambda_{25}, \lambda_{36})$, $(\lambda_{24}, \lambda_{36})$. Thus *either* S contains all of λ_{45}, λ_{25}, λ_{24}; *or* S contains λ_{36}.

The first alternative gives at once $S = S_9$. The second also gives a face-set which is inadmissible here; for it contains only four λ_{ij}, whereas $p \geq 5$, since 456 fails to combine with five forms of S.

(ii) We may now suppose that S contains one of 234, 235. Since these are equivalent under (45), we may take

$$234 \in S.$$

Thus we have now

$$(\nu, \lambda_{16}, 123, 124, 125, 126, 134, 135, 246, 256, 234) \subset S,$$

while $\lambda_{23} \notin S$, $145 \notin S$. (The set (9·26) is, of course, now accounted for.)

Now 234 fails to combine with λ_{16}, 125, 126, 135, 256, so that $p \geq 5$. It follows that $q \leq 10$ and hence that S contains at most one more μ_{ijk}. Since 125 and $126 \in S$, S contains a form from each set $(\lambda_{23}, \lambda_{14}, 235, 145), (\lambda_{23}, \lambda_{14}, 236, 146)$; since $\lambda_{23} \notin S$ and S contains at most one more μ_{ijk}, we require $\lambda_{14} \in S$, and

$$(\nu, \lambda_{14}, \lambda_{16}, 123, 124, 125, 126, 134, 135, 246, 256, 234) \subset S. \tag{9·28}$$

Suppose first that S contains no more μ_{ijk}; then $q = 9$ and $p \leqslant 6$. Since $126, 246, 256 \in S$, S contains a form from each of $(\lambda_{23}, \lambda_{15}, 236, 156)$, $(\lambda_{23}, \lambda_{45}, 236, 456)$, $(\lambda_{23}, \lambda_{46}, 235, 456)$, so that S contains all of λ_{15}, λ_{45}, λ_{46}. Also, since $246, 256 \in S$, S contains one of each set $(\lambda_{25}, \lambda_{36}, 245, 346)$, $(\lambda_{24}, \lambda_{36}, 245, 356)$, and so one of each pair $(\lambda_{25}, \lambda_{36})$, $(\lambda_{24}, \lambda_{36})$. Thus *either* $\lambda_{36} \in S$, when λ_{15} fails to combine with the seven forms $\nu, 125, 135, 246, 234, \lambda_{46}, \lambda_{36}$ of S, contradicting $p \leqslant 6$; *or* both $\lambda_{25}, \lambda_{24} \in S$, giving seven forms λ_{ij} in S and again contradicting $p \leqslant 6$.

Suppose next that S contains one more μ_{ijk}; then $q = 10$ and $p \leqslant 5$; as above, $p \geqslant 5$, so that now $p = 5$. Since both 256 and 234 fail to combine with five forms of the subset (9·28), it follows that *all further forms in S combine with both* 256 and 234. In particular we see that

$$\lambda_{15}, 145, 346, 456 \notin S.$$

Since S contains 134, 126, 246 and 256, it contains a form from each of the sets

$$(\lambda_{15}, \lambda_{36}, 145, 346), \quad (\lambda_{23}, \lambda_{15}, 236, 156), \quad (\lambda_{23}, \lambda_{45}, 235, 456), \quad (\lambda_{23}, \lambda_{46}, 235, 456);$$

hence S contains λ_{36} and a form from each pair

$$(236, 156), \quad (\lambda_{45}, 236), \quad (\lambda_{46}, 235).$$

Thus *either* S contains 236 and λ_{46}, when λ_{14} fails to combine with the six forms ν, 124, 134, 256, λ_{36}, 236 of S, contradicting $p = 5$; *or* S contains 156, λ_{45} and λ_{46}, giving $S = S'_2$.

LEMMA 9·9. *If S contains (ν, T_4) (but no subset equivalent under $\mathfrak{G}(\nu)$ to (ν, T_i) for $i<4$), then S is equivalent to one of*

$$S_{11} = (\nu, \lambda_{14}, \lambda_{16}, \lambda_{23}, \lambda_{25}, \lambda_{34}, \lambda_{36}, \lambda_{45}, T_4),$$
$$S_6 = (\nu, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{25}, \lambda_{45}, T_4).$$

Proof. We have

$$(\nu, T_4) = (\nu, 123, 124, 125, 126, 134, 135, 234, 246),$$

and $\mathfrak{G}(\nu, T_4)$ contains $\mathfrak{H} = \{(12)(34)(56), \mathfrak{R}(15)(26)\}$. Since we obtain immediately a subset (ν, T_3) if $256 \in S$, we have $256 \notin S$; applying $\mathfrak{H}$, we obtain

$$156, 256, 345, 346 \notin S.$$

Also, if both $136, 245 \in S$, S has the subset

$$(\nu, 123, 124, 125, 126, 135, 136, 245, 246) \sim (\nu, T_3),$$

under (46); using $\mathfrak{H}$, we see that S cannot contain both (136, 245), nor both (146, 235).

(i) Suppose first that $\lambda_{16} \in S, \quad \lambda_{25} \in S.$

Then $145, 236 \notin S;$

for both λ_{16}, 145 in S would give immediately a proper subset S_4 and (using $\mathfrak{H}$) both λ_{25}, 236 in S would give an equivalent subset.

Since $125 \in S$, S contains one of $(\lambda_{14}, \lambda_{26}, 145, 256)$, and hence one of $(\lambda_{14}, \lambda_{26})$. By $\mathfrak{H}$, S contains a form from each pair

$$(\lambda_{14}, \lambda_{26}), \quad (\lambda_{23}, \lambda_{15}), \quad (\lambda_{45}, \lambda_{26}), \quad (\lambda_{36}, \lambda_{15}).$$

Also, since $125 \in S$, S contains one of $(\lambda_{14}, \lambda_{23}, 145, 235)$; and, since $126 \in S$, one of $(\lambda_{14}, \lambda_{23}, 146, 236)$. Now $145 \notin S$, $236 \notin S$, and S cannot contain both 146 and 235; hence S contains one of λ_{14}, λ_{23}. Using $\mathfrak{H}$, we deduce that S contains a form from each pair

$$(\lambda_{14}, \lambda_{23}), \quad (\lambda_{45}, \lambda_{36}).$$

(*a*) Suppose now that some one of λ_{14}, λ_{23}, λ_{45}, λ_{36} does not belong to S; since these forms are equivalent under $\mathfrak{H}$, we may take

$$\lambda_{36} \notin S.$$

Then, from above, S contains λ_{15}, λ_{45} and one from each pair $(\lambda_{14}, \lambda_{26})$, $(\lambda_{14}, \lambda_{23})$. Also, since $135 \in S$, S contains one of $(\lambda_{14}, \lambda_{36}, 145, 356)$ and hence one of $(\lambda_{14}, 356)$.

Thus *either* S contains λ_{14}, giving $S = S_6$, as required; *or* S contains all of $\lambda_{26}, \lambda_{23}, 356$. This last alternative is impossible; for it gives $q \geq 9$ and $p \geq 7$, since now λ_{15} fails to combine with the seven forms $\nu, \lambda_{26}, \lambda_{23}, 125, 135, 234, 246$ of S.

(*b*) Suppose next that S contains all of λ_{14}, λ_{23}, λ_{45}, λ_{36}. Then since $\lambda_{14}, \lambda_{23} \in S$, S contains one of $(\lambda_{15}, \lambda_{34}, 156, 346)$, $(\lambda_{26}, \lambda_{34}, 256, 345)$, and hence a form from each pair $(\lambda_{15}, \lambda_{34})$, $(\lambda_{26}, \lambda_{34})$. Since $q \geq 8$, $p \leq 7$ and so S cannot now contain both λ_{15}, λ_{26}. Hence $\lambda_{34} \in S$ and we have $S = S_{11}$, as required.

(ii) Suppose next that $$\lambda_{16} \notin S, \quad \lambda_{25} \notin S.$$

Since $246, 123, 124 \in S$, S contains a form from each set

$$(\lambda_{25}, \lambda_{34}, 256, 346), \quad (\lambda_{16}, \lambda_{25}, 136, 235), \quad (\lambda_{16}, \lambda_{25}, 146, 245).$$

Thus S contains λ_{34} and a form from each pair

$$(136, 235), \quad (146, 245).$$

Since $136 \sim 235$ under $\mathfrak{H}$, we may take $136 \in S$; since S cannot contain both 136 and 245, we require $146 \in S$. But now we have $q \geq 10$ and $p \geq 6$ (since 125 fails to combine with the six forms λ_{34}, 134, 234, 246, 136, 146 of S), and this is impossible.

(iii) Suppose, finally, that S contains just one of λ_{16}, λ_{25}. Since these are equivalent under $\mathfrak{H}$, we may suppose that $$\lambda_{16} \in S, \quad \lambda_{25} \notin S.$$

We note that this condition is invariant under the subgroup

$$\mathfrak{H}' = \mathfrak{R}(16)\,(25)\,(34)$$

of H.

As in (ii), S contains one of $(\lambda_{25}, \lambda_{34}, 256, 346)$ and so

$$\lambda_{34} \in S.$$

Also, as in (i), both λ_{16}, 145 cannot belong to S and so

$$145 \notin S.$$

By lemma 9·2 (ii), S contains one of $(\lambda_{25}, 235, 256, 136, 246, 145)$ and hence one of

$$(235, 136).$$

Hence $q \geq 9$ and so $p \leq 6$. But λ_{34} fails to combine with the six forms ν, μ_{16}, 125, 126, 134, 234 of S. Hence $q = 9$ and $p = 6$. It follows that S contains just one μ_{ijk} from the pair (235, 136) and no further μ_{ijk}; and that all further forms in S must combine with λ_{34}. This last statement shows that in particular

$$\lambda_{15}, \lambda_{26} \notin S.$$

Since $125 \in S$, S contains one of $(\lambda_{14}, \lambda_{26}, 145, 256)$ and hence λ_{14}; by $\mathfrak{H}'$, $\lambda_{36} \in S$. Since $126 \in S$, S contains one of $(\lambda_{23}, \lambda_{15}, 236, 156)$ and hence λ_{23}; by $\mathfrak{H}'$, $\lambda_{45} \in S$. S thus contains the six forms λ_{16}, λ_{34}, λ_{14}, λ_{36}, λ_{23}, λ_{45}, and, since $p = 6$, S contains no further λ_{ij}. But since $126, 234 \in S$, S contains a form from each set $(\lambda_{25}, \lambda_{13}, 256, 136)$, $(\lambda_{25}, \lambda_{46}, 235, 346)$, and hence *both* 136 and 235, contradicting $q = 9$.

Thus this case is impossible, and the lemma is proved.

Lemma 9·10. *If S has a subset (ν, T_5) (but no subset equivalent under $\mathfrak{G}(\nu)$ to any (ν, T_i), $i<5$), then S is equivalent to one of*

$$\begin{aligned} S_1 &= (\nu, \lambda_{14}, \lambda_{15}, \lambda_{16}, T_5),\\ S_2'' &= (\nu, \lambda_{14}, \lambda_{23}, \lambda_{24}, \lambda_{34}, \lambda_{46}, T_5, 145, 245, 345),\\ S_3 &= (\nu, \lambda_{14}, \lambda_{15}, \lambda_{23}, \lambda_{24}, \lambda_{25}, \lambda_{34}, \lambda_{35}, T_5, 236),\\ S_6' &= (\nu, \lambda_{14}, \lambda_{15}, \lambda_{23}, \lambda_{34}, \lambda_{35}, T_5, 245),\\ S_{10} &= (\nu, \lambda_{23}, \lambda_{24}, \lambda_{25}, \lambda_{34}, \lambda_{35}, T_5, 145, 236).\end{aligned}$$

Proof. We have

$$(\nu, T_5) = (\nu, 123, 124, 125, 126, 134, 135, 136),$$

and $\mathfrak{G}(\nu, T_5)$ contains $\mathfrak{H} = \{(23), (4, 5, 6)'\}$.

If S contains both λ_{16}, 145, we have a proper subset S_4; using $\mathfrak{H}$, we therefore have

$$S \text{ cannot contain both } \lambda_{1i}, 1jk \quad (i, j, k = 4, 5, 6). \tag{9·29}$$

If S contains both 234, 246, we have a subset (ν, T_4); using $\mathfrak{H}$, we have

$$S \text{ cannot contain both } 2ij, 23i, \text{ or both } 3ij, 23i \quad (i, j = 4, 5, 6). \tag{9·30}$$

If S contains both 246, 256, we have a subset (ν, T_3); using $\mathfrak{H}$, we have

$$S \text{ cannot contain both } 2ij, 2ik, \text{ or both } 3ij, 3ik \; (i, j, k = 4, 5, 6). \tag{9·31}$$

Now by lemma 9·3 (ii), S contains a form from each set

$$(\lambda_{14}, 145, 146, 256, 356, 234), \quad (\lambda_{15}, 145, 156, 246, 346, 235), \quad (\lambda_{16}, 146, 156, 245, 345, 236). \tag{9·32}$$

I. Suppose that S contains some one of 145, 146, 156. Since these are equivalent under $\mathfrak{H}$, we may take

$$145 \in S;$$

then, by (9·29),

$$\lambda_{16} \notin S.$$

Now $(\nu, T_5, 145) \subset S$, and this subset is invariant under $\mathfrak{H}' = \{(23)(45)\}$. From (9·32), S contains one of

$$(146, 156, 245, 345, 236); \tag{9·33}$$

since, under $\mathfrak{H}'$, $146 \sim 156$ and $245 \sim 345$, there are three inequivalent choices of a form to satisfy this condition. We now consider these in turn.

(i) Suppose

$$146 \in S, \tag{9·34}$$

so that, by (9·29),

$$\lambda_{15} \notin S.$$

Also $$156 \notin S,$$

since the set $(\nu, 1ij)$ $(2 \leqslant i < j \leqslant 6)$ determines a face. The subset $(\nu, T_5, 145, 146)$ is invariant under the group $\{(2, 3, 4)', (5, 6)\}$.

Since $125, 126 \in S$, S contains a form from each set $(\lambda_{16}, \lambda_{24}, 156, 245)$, $(\lambda_{15}, \lambda_{24}, 156, 246)$, and hence a form from each pair $(\lambda_{24}, 245)$, $(\lambda_{24}, 246)$. By (9·31), S cannot contain both 245, 246, and so we require $\lambda_{24} \in S$. Applying $(2, 3, 4)'$, we see that therefore

$$\lambda_{23}, \lambda_{24}, \lambda_{34} \in S.$$

Since $\lambda_{23} \in S$, S contains a form from the set $(\lambda_{25}, \lambda_{36}, 245, 346)$. Applying the group $\{(2, 3, 4)', (5, 6)\}$, we see that S contains a form from each of the six sets

$$(\lambda_{i5}, \lambda_{j6}, ik5, jk6) \quad (i, j, k = 2, 3, 4). \tag{9·35}$$

Now every form occurring in the sets (9·35) fails to combine with six forms of the known subset $(\nu, \lambda_{23}, \lambda_{24}, \lambda_{34}, T_5, 145, 146)$ of S; to see this, it is necessary only to check for λ_{25} and 235, since

$$\lambda_{25} \sim \lambda_{35} \sim \lambda_{45} \sim \lambda_{26} \sim \lambda_{36} \sim \lambda_{46},$$

$$235 \sim 245 \sim 345 \sim 236 \sim 246 \sim 256$$

under $\{(2, 3, 4)', (5, 6)\}$. Hence $p \geqslant 6$ and so $q \leqslant 9$; but S already contains nine μ_{ijk}, so that $p = 6$, $q = 9$ and S contains no further μ_{ijk}. From the sets (9·35), it follows that S contains a form from every pair

$$(\lambda_{i5}, \lambda_{j6}) \quad (i, j = 2, 3, 4).$$

As above, all λ occurring here are equivalent; we may suppose then that

$$\lambda_{25} \in S.$$

Since $p = 6$, all further λ_{ij} in S must combine with λ_{25} and so $\lambda_{36} \notin S$, $\lambda_{46} \notin S$; hence $\lambda_{35} \in S$, $\lambda_{45} \in S$. This is impossible, since now S has the proper subset

$$(\lambda_{23}, \lambda_{24}, \lambda_{25}, \lambda_{34}, \lambda_{35}, \lambda_{45}, 123, 124, 125, 134, 135, 145)$$

which determines a face.

(ii) Suppose next that $$245 \in S,$$

where, after (i), we have $$146, 156 \notin S.$$

Also, by (9·30) and (9·31), $$234, 235, 246, 256 \notin S.$$

We now have $$(\nu, T_5, 145, 245) \subset S,$$

and this subset is invariant under $\mathfrak{H}'' = \{(45)\}$.

Since $124 \in S$, S contains one of $(\lambda_{16}, \lambda_{23}, 146, 234)$ and so

$$\lambda_{23} \in S.$$

Now S cannot contain both λ_{14}, λ_{15}, since, by lemma 9·2, S cannot have

$$(145, \lambda_{14}, \lambda_{15}, 123, 136)$$

as a subset. Since $\lambda_{14} \sim \lambda_{15}$ under (45), we may without loss assume that

$$\lambda_{15} \notin S.$$

Since $126 \in S$, S contains a form from $(\lambda_{15}, \lambda_{24}, 156, 246)$, and so

$$\lambda_{24} \in S.$$

Since $136, 135 \in S$, S contains a form from each set $(\lambda_{15}, \lambda_{34}, 156, 346)$, $(\lambda_{16}, \lambda_{34}, 156, 345)$, and hence one from each pair $(\lambda_{34}, 346)$, $(\lambda_{34}, 345)$. By (9·31), S cannot contain both 345, 346, and so

$$\lambda_{34} \in S.$$

Now
$$\lambda_{34} \notin S,$$

since otherwise S would have as a proper subset the face-set

$$(\nu, \lambda_{23}, \lambda_{34}, \lambda_{35}, 123, 124, 125, 134, 135, 136, 145, 245).$$

Since $134 \in S$, S contains one of $(\lambda_{16}, \lambda_{35}, 146, 345)$, whence now

$$345 \in S.$$

Now all our conditions are invariant under (23), whence it follows that

$$\lambda_{24} \in S, \quad \lambda_{25} \notin S.$$

Finally, since $\lambda_{34} \in S$, S contains one of $(\lambda_{35}, \lambda_{46}, 235, 246)$ and so

$$\lambda_{46} \in S;$$

and since, $126 \in S$, S contains one of $(\nu_{14}, \lambda_{25}, 146, 256)$ and so

$$\lambda_{14} \in S.$$

Collecting these results, we see that now $S = S_2'''$, as required.

(iii) Suppose finally that, to satisfy (9·33), we have

$$236 \in S.$$

Then, by (9·30),
$$246, 256, 346, 356 \notin S.$$

Also, by (i) and (ii), we may now take

$$146, \ 156, \ 245, \ 345 \notin S.$$

Our conditions are still invariant under $\mathfrak{H}' = \{(23), (45)\}$.

Since $125 \in S$, S contains one of $(\lambda_{16}, \lambda_{24}, 156, 245)$, so that $\lambda_{24} \in S$. Applying $\mathfrak{H}'$, we have

$$\lambda_{24}, \lambda_{25}, \lambda_{34}, \lambda_{35} \in S.$$

Since $125 \in S$, S contains one of $(\lambda_{16}, \lambda_{23}, 156, 235)$ and so one of $(\lambda_{23}, 235)$. Applying $\mathfrak{H}'$, we see that S contains a form of each pair

$$(\lambda_{23}, 235), \quad (\lambda_{23}, 234).$$

It is easy to see that not both 234, 235 can belong to S (since this would give $q \geq 11, p \geq 6$). Hence $\lambda_{23} \in S$ and $S = S_{10}$, as required.

II. We may now suppose, after I, that

$$145, 146, 156 \notin S.$$

The sets (9·32) now reduce to

$$(\lambda_{14}, 256, 356, 234), \quad (\lambda_{15}, 246, 346, 235), \quad (\lambda_{16}, 245, 345, 236). \tag{9·36}$$

We note also that if all three λ_{ij} occurring here belong to S, then $S = S_1$, as required. Thus hereafter we may suppose that S does not contain all of λ_{14}, λ_{15}, λ_{16} (these forms being equivalent under $\mathfrak{H}$), and we split cases according to the number of them in S.

(i) $$\lambda_{14}, \lambda_{15} \in S, \quad \lambda_{16} \notin S;$$

this condition is invariant under $\mathfrak{H}' = \{(23), (45)\}$. From (9·36), S contains one of

$$(245, 345, 236).$$

Since $124 \in S$, S contains a form from each set $(\lambda_{16}, \lambda_{23}, 146, 234)$, $(\lambda_{16}, \lambda_{25}, 146, 245)$, and hence from $(\lambda_{23}, 234)$, $(\lambda_{25}, 245)$. Applying $\mathfrak{H}'$, we see that S contains a form from each pair

$$(\lambda_{23}, 234), \quad (\lambda_{23}, 235), \quad (\lambda_{24}, 245), \quad (\lambda_{25}, 245), \quad (\lambda_{34}, 345), \quad (\lambda_{35}, 345). \qquad (9·37)$$

(*a*) If neither 245, 345 belong to S, then clearly

$$236, \lambda_{24}, \lambda_{25}, \lambda_{34}, \lambda_{35} \in S.$$

It is clearly impossible that both 234, 235 belong to S (else $q \geqslant 10$, $p \geqslant 6$), whence $\lambda_{23} \in S$. Now $S = S_3$, as required.

(*b*) Otherwise, one of the (equivalent) forms 245, 345 belongs to S, say

$$245 \in S.$$

By (9·30) and (9·31), $$234, 235, 246, 256 \notin S,$$

and so, from (9·37), $$\lambda_{23} \in S$$

and S contains a form from each pair $(\lambda_{34}, 345)$, $(\lambda_{35}, 345)$.

If now $345 \notin S$, we require $\lambda_{34} \in S$, $\lambda_{35} \in S$, and so $S = S_6'$, as required. Otherwise we have

$$345 \in S$$

and so $$346, 356 \notin S.$$

Our conditions are now again invariant under $\mathfrak{H}' = \{(23), (45)\}$.

Since $\lambda_{23} \in S$, S contains a form from each set $(\lambda_{24}, \lambda_{35}, 246, 356)$, $(\lambda_{25}, \lambda_{34}, 256, 346)$, and hence a form from each pair $$(\lambda_{24}, \lambda_{35}), \quad (\lambda_{25}, \lambda_{34}).$$

Since $\lambda_{25} \sim \lambda_{34}$ under $\mathfrak{H}'$, we may take $$\lambda_{34} \in S.$$

Then $$\lambda_{35} \notin S$$

(since S cannot have S_6' as a proper subset), and so

$$\lambda_{24} \in S.$$

We now have $$(\nu, \lambda_{14}, \lambda_{15}, \lambda_{23}, \lambda_{24}, \lambda_{34}, T_5, 245, 345) \subset S,$$

and each of λ_{15}, λ_{24}, λ_{34} fails to combine with six forms of this subset. Hence $p \geqslant 6$; since $q \geqslant 9$, it follows that $p = 6$, $q = 9$ and S contains just one more λ_{ij}. This λ_{ij} must combine with λ_{15}, λ_{24} and λ_{34} (since $p = 6$), and therefore can only be λ_{45}. Thus $\lambda_{45} \in S$. But now λ_{45} fails to combine with the seven forms ν, λ_{23}, 123, 126, 136, 245, 345 of S, contradicting $p = 6$.

(ii) Suppose finally that some two of $\lambda_{14}, \lambda_{15}, \lambda_{16}$ do not belong to S. We may take

$$\lambda_{14}, \lambda_{15} \notin S,$$

and our conditions are again invariant under $\mathfrak{H}' = \{(23), (45)\}$. From the first two sets (9·36), we see that S contains a form from each of the sets (256, 356, 234), (246, 346, 235).

(*a*) If $234 \in S$, then, by (9·30), $246 \notin S$, $346 \notin S$, whence $235 \in S$. Thus, using (9·30), we now have

$$234, 235 \in S,$$

$$245, 246, 256, 345, 346, 356 \notin S.$$

Since $124, 126 \in S$, S contains a form from each set $(\lambda_{15}, \lambda_{26}, 145, 246)$, $(\lambda_{14}, \lambda_{25}, 146, 256)$, and so $\lambda_{26} \in S$, $\lambda_{25} \in S$. Using $\mathfrak{H}'$, we obtain

$$\lambda_{24}, \lambda_{25}, \lambda_{26}, \lambda_{34}, \lambda_{35}, \lambda_{36} \in S.$$

We have now found sixteen forms in S, so that S can contain no further forms, in particular none from the last set (9·36); a contradiction.

(*b*) If $234 \notin S$, S contains one of 256, 356, and these are equivalent under $\mathfrak{H}'$. We may therefore take

$$356 \in S$$

Then $235, 346 \notin S$, and so, from the second set (9·36),

$$246 \in S.$$

Thus now
$$234, 235, 236, 345, 346, 245, 256 \notin S$$

(using (9·30), (9·31)); from the third set (9·36) we have at once

$$\lambda_{16} \in S.$$

We note that our conditions are invariant under $\mathfrak{H}'' = \{(23)\,(45)\} \subset \mathfrak{H}'$.

Since $126 \in S$, S contains one of $(\lambda_{14}, \lambda_{25}, 146, 256)$ and so λ_{25}. By $\mathfrak{H}''$,

$$\lambda_{25}, \lambda_{34} \in S.$$

Since $124 \in S$, S contains one of $(\lambda_{15}, \lambda_{23}, 145, 234)$ and so

$$\lambda_{23} \in S.$$

Also, since $125 \in S$, S contains one of $(\lambda_{14}, \lambda_{26}, 145, 256)$ and so λ_{26}. By $\mathfrak{H}''$,

$$\lambda_{26}, \lambda_{36} \in S.$$

But now we have $q \geqslant 9$, while λ_{25} fails to combine with the seven forms

$$\nu, \lambda_{34}, \lambda_{36}, \lambda_{16}, 125, 134, 136$$

of S, giving $p \geqslant 7$; this is impossible.

LEMMA 9·11. *If S has a subset (ν, T_6) (but no subset equivalent under $\mathfrak{G}(\nu)$ to any (ν, T_i) $(i<6)$), then S is equivalent to one of*

$$S_4 = (\nu, \lambda_{16}, T_6),$$
$$S_5 = (\nu, \lambda_{23}, \lambda_{24}, \lambda_{25}, T_6, 126, 345).$$

Proof. We have
$$(\nu, T_6) = (\nu, 123, 124, 125, 134, 135, 145),$$
and $\mathfrak{G}(\nu, T_6)$ contains $\mathfrak{H} = \{(2, 3, 4, 5)', \mathfrak{R}(16)\}$.

We observe first that if $\lambda_{16} \in S$, then $S = S_4$, as required.

Henceforward then we may suppose that

$$\lambda_{16} \notin S.$$

Next, if $126, 136 \in S$, we have a subset (ν, T_7); applying $\mathfrak{H}$, we obtain:

$$S \text{ contains at most one of } 126, 136, 146, 156; \tag{9·38}$$

$$S \text{ contains at most one of } 234, 235, 245, 345. \tag{9·39}$$

By lemma 9·2 (ii), S contains a form from every set

$$(\lambda_{16}, 16i, 16j, ijk, ijl, kl6) \quad (i, j, k, l = 2, 3, 4, 5). \tag{9·40}$$

If now S contains none of the eight forms listed in (9·38), (9·39), it follows that S contains all six forms $kl6$ $(k, l = 2, 3, 4, 5)$; this is trivially impossible (e.g. it gives a subset (ν, T_1) of S). Since these eight forms are equivalent under $\mathfrak{H}$, we may suppose that

$$126 \in S;$$

then, by (9·38), $\qquad 136, 146, 156 \notin S.$

(i) Suppose first that $\qquad 345 \in S.$

Then, by (9·39), $\qquad 234, 235, 245 \notin S.$

(We note that all six sets (9·40) are now accounted for.) Our assumptions $126 \in S$, $345 \in S$ are invariant under the subgroup

$$\mathfrak{H}' = \{(3, 4, 5)', \mathfrak{R}(16)\}$$

of $\mathfrak{H}$, under which $\qquad \lambda_{23} \sim \lambda_{24} \sim \lambda_{25}.$

Since $123 \in S$, S contains one of $(\lambda_{16}, \lambda_{24}, 136, 234)$ and so $\lambda_{24} \in S$. It follows that all of $\lambda_{23}, \lambda_{24}, \lambda_{25}$ belong to S, and $S = S_5$, as required.

(ii) Suppose next that $\qquad 345 \notin S.$

Our conditions are now invariant under $\mathfrak{H}'' = \{(3, 4, 5)'\}$. From (9·40) we find that S contains a form from each of the pairs

$$(234, 256), \quad (235, 246), \quad (245, 236)$$

(after dropping the excluded forms λ_{16}, 136, 146, 156, 345). Since S contains at most one of 234, 235, 245, and these forms are equivalent under $\mathfrak{H}''$, we see that there are just two inequivalent cases to consider:

(*a*) S contains 236, 246, 256;

(*b*) S contains 245, 246, 256.

In either case S has a subset (ν, T_3), contrary to assumption.

We have now shown, as asserted above lemma 9·6, that all face-sets S are equivalent to one of the sixteen sets $S_1, \ldots, S_{11}, S'_2, \ldots, S'_{10}$ listed there. Our next step is to establish the equivalence of the last five of these sets, $S'_2, \ldots, S'_{10}$, with the corresponding unprimed sets.

We could of course write down a particular element of $\mathfrak{G}$ effecting the required transformation in each case. A less synthetic method is to use the following property of $\mathfrak{G}$:

LEMMA 9·12. *If* $(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5)$ *is a set of five forms, every pair of which combines, with the exception of* κ_4, κ_5, *then there exists a unique* element* $\mathfrak{T}$ *of* $\mathfrak{G}$ *with*

$$\mathfrak{T}(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5) = (\nu, 123, 124, 125, 134). \tag{9·41}$$

Proof. By lemma 9·1 (iii), we can find a $\mathfrak{T}$ satisfying

$$\mathfrak{T}(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (\nu, 123, 124, 125).$$

Then $T(\kappa_5)$ must combine with ν, 123, 124 but not with 125, and is therefore easily seen to be either 134 or 234. Replacing $\mathfrak{T}$ by (12) $\mathfrak{T}$ if necessary, we have (9·41) satisfied.

To prove uniqueness, we have to show that if $\mathfrak{T}$ leaves each of the forms ν, 123, 124, 125, 134 invariant, then $\mathfrak{T} = \pm\mathfrak{J}$. Now $\mathfrak{T}(126) = 126$, since 126 is the unique form combining with all of ν, 123, 124, 125. Also if we fix the sign of T by taking $\mathfrak{T}(\nu) = +\nu$, then the signs of $\mathfrak{T}(123), \ldots, \mathfrak{T}(126), \mathfrak{T}(134)$, are determined. Since the six forms $\nu, 123, \ldots, 126, 134$ are easily seen to be linearly independent, it therefore follows that $\mathfrak{T} = \mathfrak{J}$, as required.

We now prove

LEMMA 9·13. *Under* $\mathfrak{G}$, *we have*

$$S_2 \sim S_2' \sim S_2'', \quad S_5 \sim S_5', \quad S_6 \sim S_6', \quad S_{10} \sim S_{10}'.$$

Proof. In each case we establish a suitable transformation by lemma 9·12, using appropriate linear relations between forms to find the complete map of each set. It will be necessary to give all the details in one case only, say $S_2 \sim S_2'$.

(i) We have, after reordering,

$$S_2' = (\nu, 123, 124, 125, 126, 134, 135, 156, 234, 246, 256, \lambda_{14}, \lambda_{16}, \lambda_{36}, \lambda_{45}, \lambda_{46}),$$
$$S_2 = (123, \lambda_{15}, 134, \lambda_{16}, 124, \lambda_{35}, 456, \lambda_{26}, 136, \nu, 125, 356, 256, 245, 346, \lambda_{23}).$$

By lemma 9·12, a uniquely determined element $\mathfrak{T}$ of $\mathfrak{G}$ satisfies

$$\mathfrak{T}(\nu, 123, 124, 125, 134) = (123, \lambda_{15}, 134, \lambda_{16}, \lambda_{35}).$$

We now show that $\mathfrak{T}(S_2') = S_2$, in the order specified above.

First, $\mathfrak{T}(126) = 124$ (as the unique form combining with all of $123, \lambda_{15}, 134, \lambda_{16}$). From the linear relations (with suitable sign!)

$$\lambda_{36} = 123 - 126, \quad \lambda_{45} = 124 - 125, \quad \lambda_{46} = 124 - 126,$$

we then obtain

$$\mathfrak{T}(\lambda_{36}) = \lambda_{15} - 124 = 245,$$
$$\mathfrak{T}(\lambda_{45}) = 134 - \lambda_{16} = 346,$$
$$\mathfrak{T}(\lambda_{46}) = 134 - 124 = \lambda_{23},$$

It now follows that $\mathfrak{T}(135) = 456, \quad \mathfrak{T}(156) = \lambda_{26}$,

by using the relations $135 = 134 + \lambda_{45}, \quad 156 = 135 + \lambda_{36}$;

and then that $\mathfrak{T}(234) = 136, \quad \mathfrak{T}(246) = \nu, \quad \mathfrak{T}(256) = 125$

from the relations $234 = \nu - 156, \quad 246 = \nu - 135, \quad 256 = \nu - 134$.

* $\mathfrak{T}$ is unique in so far as we are identifying transformations with their negatives.

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Finally, we have $\mathfrak{T}(\lambda_{14}) = \mathfrak{T}(123-234) = \lambda_{15}-136 = 356,$

$\mathfrak{T}(\lambda_{16}) = \mathfrak{T}(124-246) = 134-\nu = 256.$

(ii) The remaining four equivalences may be established similarly, beginning with the subset $(\nu, 123, 124, 125, 134)$ in the first set of each pair; it suffices to write down an appropriate ordering:

$$S_2''' = (\nu, 123, 124, 125, 126, 134, 135, 136, 145, 245, 345, \lambda_{14}, \lambda_{23}, \lambda_{24}, \lambda_{34}, \lambda_{46}),$$
$$S_2 = (\lambda_{15}, \lambda_{16}, 134, 123, 124, 256, 456, 356, \lambda_{35}, 136, 245, \lambda_{26}, \nu, 125, 346, \lambda_{23});$$
$$S_5' = (\nu, 123, 124, 125, 126, 134, 135, 246, 256, \lambda_{16}, \lambda_{23}, \lambda_{36}),$$
$$S_5 = (\lambda_{23}, 345, \lambda_{24}, \lambda_{25}, 126, 134, 135, 125, 124, 145, 123, \nu);$$
$$S_6' = (\nu, 123, 124, 125, 126, 134, 135, 234, 246, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{25}, \lambda_{45}),$$
$$S_6 = (\lambda_{15}, 123, 136, 126, \lambda_{14}, 134, 124, \lambda_{35}, 245, 125, 135, \nu, \lambda_{34}, \lambda_{23});$$
$$S_{10}' = (\nu, 123, 124, 125, 126, 134, 135, 246, 256, 346, \lambda_{14}, \lambda_{16}, \lambda_{23}, \lambda_{24}, \lambda_{45}),$$
$$S_{10} = (\lambda_{25}, 126, \lambda_{24}, 123, 236, \lambda_{23}, 124, 145, \lambda_{35}, 135, \nu, 125, \lambda_{34}, 136, 134).$$

Lemma 9·13 shows now that all face-sets are equivalent to one of $S_1, \ldots, S_{11}$. To establish that these form a complete system of representatives, we have only to prove

LEMMA 9·14. *No two sets $S_1, \ldots, S_{11}$ are equivalent.*

Proof. Clearly equivalent sets must contain the same number of forms and have the same value of p (defined, as in lemma 9·5 as the maximum number of forms of a set not combining with one form of the set). For the sets $S_1, \ldots, S_{11}$, p is simply the number of λ_{ij} occurring.

An inspection of the sets shows that the only possible equivalences are between:

$$S_3(16), \quad S_7(16), \quad S_{11}(16) \quad (\text{with } p = 7);$$
$$S_5(12), \quad S_8(12) \quad (\text{with } p = 3).$$

(i) S_3 contains a form which combines with all but one other form of the sets, namely, 123, which fails only to combine with λ_{23}. S_7 and S_{11} are easily seen not to have this property, which is clearly invariant under equivalence transformations. Hence

$$S_3 \nsim S_7, \quad S_3 \nsim S_{11}.$$

(ii) S_7 has only one form, namely, ν, which fails to combine with seven forms of the set. S_{11} does not have this property; e.g. both ν and λ_{16} fail to combine with seven forms of S_{11}. Hence

$$S_7 \nsim S_{11}.$$

(iii) In S_8, ν fails to combine with $\lambda_{16}, \lambda_{25}, \lambda_{34}$, and no two of these three forms combine. S_5 has no form with this property; to see this, it suffices to test for $\nu, \lambda_{23}, 123, 126$ since every form of S_5 is equivalent to one of these under the group $\{(3, 4, 5)', \mathfrak{R}(16)\} \subset \mathfrak{G}(S_5)$. Hence

$$S_5 \nsim S_8.$$

We have now established the 11 inequivalent classes of faces of $R(\phi_2)$, in agreement with table 2, and it remains for us to consider the neighbours of ϕ_2 along these faces. As we shall see, it is necessary only to prove

LEMMA 9·15. *The neighbours of ϕ_2 along the faces W_4, W_5, W_6 (determined by the face-sets S_4, S_5, S_6) are equivalent to ϕ_2.*

Proof. Probably the simplest method of obtaining the quadratic form $\psi_i(\mathbf{x})$ from the face set S_i is as follows:

(*a*) Solve the equations (9·13) corresponding to the forms not in S_i; (*b*) with an appropriate sign, this yields

$$\psi_i(\mathbf{y}) = \Sigma p_{ij} y_i y_j.$$

(*c*) Apply the transformation (9·2), i.e.

$$y_1 = \sum_1^6 x_i, \quad y_2 = x_1 - x_2, \quad y_3 = x_1 - x_3, \quad y_i = -x_i \quad (i = 4, 5, 6),$$

which gives $\psi_i(\mathbf{x})$.

(i) In place of S_4 we take

$$S_4' = (\nu, \lambda_{24}, 134, 145, 146, 345, 346, 456),$$

obtained by the transformation (14) (26). This gives

$$\psi_4(\mathbf{x}) = -x_2 x_4.$$

Now $$f_4(\mathbf{x}) = \phi_2(\mathbf{x}) + \psi_4(\mathbf{x}) = \phi_0(\mathbf{x}) - x_1x_2 - x_1x_3 - x_2x_4 \sim \phi_2(\mathbf{x}),$$

as is essentially shown in § 4 (vi).

(ii) From S_5, we obtain

$$\psi_5(\mathbf{x}) = x_1^2 + x_1x_3 + x_1x_4 + x_1x_5 + x_1x_6 - x_2x_3 - x_2x_4 - x_2x_5,$$

$$f_5(\mathbf{x}) = \phi_2(\mathbf{x}) + \psi_5(\mathbf{x}) = 2x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_1x_3 + 2x_1x_4 + 2x_1x_5 + 2x_1x_6 + x_2x_6 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6.$$

This is equivalent to $\phi_2(\mathbf{x})$ under the transformation

$$\mathbf{x} \to \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & 1 & \cdot & \cdot & \cdot & 1 \end{pmatrix} \mathbf{x}.$$

(iii) From S_6 we obtain

$$\psi_6(\mathbf{x}) = 2x_1^2 + x_4^2 + x_5^2 + x_6^2 + x_1x_2 + 2x_1x_3 + 2x_1x_4 + 3x_1x_5 + 3x_1x_6 + x_2x_4 + x_2x_6 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_5 + 2x_4x_6 + 2x_5x_6,$$

$$f_6(\mathbf{x}) = \phi_2(\mathbf{x}) + \psi_6(\mathbf{x})$$
$$= 3x_1^2 + x_2^2 + x_3^2 + 2x_4^2 + 2x_5^2 + 2x_6^2 + x_1x_2 + 2x_1x_3 + 3x_1x_4 + 4x_1x_5 + 4x_1x_6 + x_2x_3 + 2x_2x_4 + x_2x_5 + 2x_2x_6 + 2x_3x_4 + 2x_3x_5 + 2x_3x_6 + 2x_4x_5 + 3x_4x_6 + 3x_5x_6.$$

This is equivalent to $\phi_2(\mathbf{x})$ under the transformation

$$\mathbf{x} \to \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 1 & \cdot & 1 & 1 & 1 \\ 1 & \cdot & 1 & 1 & \cdot & 1 \\ -1 & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ -2 & \cdot & \cdot & \cdot & -1 & -1 \end{pmatrix} \mathbf{x}.$$

This completes the proof of the lemma.

There now remain eight inequivalent faces of $R(\phi_2)$ to consider. Since table 2 has now been verified for all forms ϕ_i except ϕ_2, it is easy to see that $R(\phi_2)$ must have eight inequivalent faces with neighbours ϕ_1, ϕ_1, ϕ_1, ϕ_3, ϕ_4, ϕ_5, ϕ_5, ϕ_6, and with the numbers of edges as given in the table. Thus, for example, ϕ_1 has three inequivalent faces $W_3(25)$, $W_4(20)$, $W_5(20)$ with neighbour (equivalent to) ϕ_2; hence ϕ_2 must have three inequivalent faces $W(25)$, $W(20)$, $W(20)$ with neighbour (equivalent to) ϕ_1; these appear in table 2 as $W_1(25)$, $W_2(20)$, $W_3(20)$. As an extra check, we see that the number, t, of edges of these eight faces, as given in table 2, agrees with the number, $36-t$, of forms in the corresponding sets* S_i.

Thus table 2 is now completely verified for ϕ_2.

10. Conclusion

We have now completed the program outlined in § 2 by finding all neighbours of the inequivalent faces of each $R(\phi_i)$ $(i = 0, \ldots, 6)$. Moreover, we have established at the beginning of each section that each ϕ_i is perfect, and that each is extreme (with the exception of ϕ_5, which is not extreme). The proof of theorem 1 is therefore complete.

It is unfortunate that the analysis of § 9 is so long and detailed, since this leaves little hope that these methods can be practicable for any $n > 6$. There is every reason to believe that the number of extreme (or merely perfect) forms increases rapidly with n, and that very large values of $s-N$ occur (e.g. for the known absolutely extreme forms in seven and eight variables); the total number of inequivalent faces of the various regions $R(\phi)$ will therefore be overwhelmingly large.

Voronoi's algorithm has however the desirable property of providing a check on the complete analysis, as was indicated at the end of § 9; every common face (or, rather, class of equivalent faces) of regions $R(\phi_i)$, $R(\phi_j)$ with $i \neq j$ is found just twice.

The two new forms ϕ_5 and ϕ_6 produced by the analysis suggest some interesting generalizations to extreme forms in higher dimensions, which the author hopes to consider in a separate article. Thus our representation of ϕ_6 (§ 7) generalizes to the n-dimensional form

$$\phi = \sum_0^n y_i^2,$$

with

$$\sum_0^n y_i = 0, \quad \sum_0^n i y_i \equiv 0 \pmod{n+1},$$

which is almost certainly extreme for all $n \geqslant 6$

I should like, in conclusion, to express my gratitude to Professor T. G. Room, F.R.S., for helpful discussions on the geometrical and group-theoretical aspects of the forms ϕ_2 and ϕ_4; and to Mr W. B. Smith-White for his comments on Voronoi's papers.

* The correspondence between faces W and face-sets S_i is not yet quite complete. Thus although there must be two faces $W(20)$ with neighbour ϕ_1, we do not yet know whether they arise from S_2, S_3, S_7, S_9 or S_{11}, all of which have $36-t = 16$. This is irrelevant from the point of view of table 2; however, the author has verified the correspondence directly, by finding the explicit form for each neighbour.

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Appendix

Baker's table of synthemes referred to on p. 483, footnote, is:

	1	2	3	4	5	6
1	—	14.25.36	16.24.35	13.26.45	12.34.56	15.23.46
2	14.25.36	—	15.26.34	12.35.46	16.23.45	13.24.56
3	16.24.35	15.26.34	—	14.23.56	13.25.46	12.36.45
4	13.26.45	12.35.46	14.23.56	—	15.24.36	16.25.34
5	12.34.56	16.23.45	13.25.46	15.24.36	—	14.26.35
6	15.23.46	13.24.56	12.36.45	16.25.34	14.26.35	—

This (symmetrical) table has the property that each of the fifteen duads ij (unordered pairs, with $i \neq j$, chosen from $1, \ldots, 6$) occurs just once in each row and column. Hence (9·12) is satisfied, in terms of fifteen parameters $u_{ij} = u_{ji}$ $(i \neq j)$, by taking

$$\gamma = \Sigma u_{ij} \quad \text{and} \quad \alpha_{ij} = u_{ab} + u_{cd} + u_{ef},$$

where $ab.\,cd.\,ef$ is the syntheme in position (i, j) in the above table.

The complete solution of the equations (9·9), (9·10) may now be constructed as follows:

For each triple ijk, let pqr be the complementary triple. The six synthemes in positions (i,j), (i,k), (j,k), (p,q), (p,r), (q,r) are found to contain all nine duads formed by taking one element from each of two other complementary triples abd, def. We then have

$$\beta_{ijk} = -u_{ab} - u_{ac} - u_{bc}, \quad \beta_{pqr} = -u_{de} - u_{df} - u_{ef}$$

(or conversely). Defining the β_{ijk} successively so that two β's have a common element u_{ab} if and only if they have one common suffix, we obtain the solution

$$\begin{aligned}
\beta_{123} &= -u_{12} - u_{13} - u_{23}, & \beta_{456} &= -u_{45} - u_{46} - u_{56},\\
\beta_{124} &= -u_{15} - u_{16} - u_{56}, & \beta_{356} &= -u_{23} - u_{24} - u_{34},\\
\beta_{125} &= -u_{24} - u_{26} - u_{46}, & \beta_{346} &= -u_{13} - u_{15} - u_{35},\\
\beta_{126} &= -u_{34} - u_{35} - u_{45}, & \beta_{345} &= -u_{12} - u_{16} - u_{26},\\
\beta_{134} &= -u_{34} - u_{36} - u_{46}, & \beta_{256} &= -u_{12} - u_{15} - u_{25},\\
\beta_{135} &= -u_{14} - u_{15} - u_{45}, & \beta_{246} &= -u_{23} - u_{26} - u_{36},\\
\beta_{136} &= -u_{25} - u_{26} - u_{56}, & \beta_{245} &= -u_{13} - u_{14} - u_{34},\\
\beta_{145} &= -u_{23} - u_{25} - u_{35}, & \beta_{236} &= -u_{14} - u_{16} - u_{46},\\
\beta_{146} &= -u_{12} - u_{14} - u_{24}, & \beta_{235} &= -u_{35} - u_{36} - u_{56},\\
\beta_{156} &= -u_{13} - u_{16} - u_{36}, & \beta_{234} &= -u_{24} - u_{25} - u_{45}.
\end{aligned}$$